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# LOWER AND UPPER LOEB-INTEGRALS

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Abstract. We introduce the concepts of lower and upper Loeb-integrals for an internal integration structure. These are concepts which are similarly useful for Loebs internal integration theory as the concepts of inner and outer Loeb-measures for Loebs measure theory.

## 1. Introduction and notation

Almost the whole nonstandard measure and probability theory is based on fundamental concepts and results of Loeb. Starting from an internal content, Loeb constructed in particular an important standard measure, called Loeb-measure in the literature (see [8], [9]). This Loeb-measure has been investigated and applied in many papers by Loeb and other authors to obtain new results in various fields of mathematics such as e.g. in mathematical physics and economics, in measure and probability theory and in potential theory. Loeb extended his measure theoretical approach to integration theory and introduced the so-called Loeb-integrals (see [10], [11]).

To construct the Loeb-measure, Loeb used the Carathéodory extension theorem. Another construction of the Loeb-measure for finite internal contents can be given in terms of the inner and outer Loeb-measures. Sommers (see [13]) investigated inner Loeb-measures also for nonfinite internal contents.

Inner and outer Loeb-measures are very powerful concepts that can be used in many situations. Inner and outer Loeb-measures have been applied e.g. in the construction of Radon measures and $\tau$-smooth measures, in the extension of $\tau$-smooth Baire or Radon-Baire measures to $\tau$-smooth Borel or Radon-Borel measures, or in compactness criteria for families of probability measures with respect to the weak topology (see [5], [6]).

To develop the Loeb integration theory for internal integration structures we introduce a lower and an upper Loeb-integral; concepts which are of comparable interest and usefulness as inner and outer Loeb-measures. A similar concept of lower Loeb-integrals can be found in Aldaz (see [1]).

In the following we consider a sufficiently rich superstructure $\widehat{S}$ and work with a nonstandard model for this superstructure, which is polysaturated, i.e. if $\mathcal{C}$ is a system of internal sets with cardinality $|\mathcal{C}|$ smaller than or equal to the cardinality $|\widehat{S}|$, then we have $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$ if $\mathcal{C}$ has the finite intersection property. For the general theory of nonstandard-analysis see the books of Cutland [2], Hurd-Loeb [4]

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or [7]. Let $fin(^*\mathbb{R})$ be the set of finite numbers of $^*\mathbb{R}$. For $a \in fin(^*\mathbb{R})$, $^\circ a$ denotes the standard number nearest to $a$ in $\mathbb{R}$. If $a \in {}^*\mathbb{R}$ is negative (positive) infinite put $^\circ a = -\infty$ $(\infty)$.

In the following, let $\mathcal{E} \subset {}^*\mathbb{R}^Y$ be an internal Stonian lattice and let $i : \mathcal{E} \to {}^*\mathbb{R}$ be an internal, positive linear functional, i.e. $i$ is internal and for all $e, e_1, e_2 \in \mathcal{E}$ and $\alpha_1, \alpha_2 \in {}^*\mathbb{R}$

$$|e| \in \mathcal{E}, 1 \wedge e \in \mathcal{E} \text{ and } \alpha_1 e_1 + \alpha_2 e_2 \in \mathcal{E}; \tag{1}$$

$$i(\alpha_1 e_1 + \alpha_2 e_2) = \alpha_1 i(e_1) + \alpha_2 i(e_2); \tag{2}$$

$$e \geq 0 \Rightarrow i(e) \geq 0. \tag{3}$$

We call $(\mathcal{E}, i)$ an *internal integration structure* on $Y$ (see Hurd-Loeb [4], pp. 166–168). Let

$$\mathcal{E}^{fin} := \{e \in \mathcal{E} : i(|e|) \in fin(^*\mathbb{R})\}$$

and put, for $g : Y \to {}^*\mathbb{R}$,

$$\underline{i}(g) := \sup\{^\circ i(e) : \mathcal{E}^{fin} \ni e \leq g\};$$
$$\overline{i}(g) := \inf\{^\circ i(e) : g \leq e \in \mathcal{E}^{fin}\};$$
$$\mathcal{L}(i) := \{f \in \mathbb{R}^Y : \underline{i}(f) = \overline{i}(f) \in \mathbb{R}\}.$$

$\underline{i}$ is called the *lower Loeb-integral,* $\overline{i}$ the *upper Loeb-integral* and $\mathcal{L}(i)$ the system of *Loeb-integrable functions.* If $f \in \mathcal{L}(i)$, we write $i^L(f)$ instead of $\underline{i}(f)$ $(= \overline{i}(f))$, and call $i^L(f)$ the *Loeb-integral* of $f$.

Let $\rho : \mathcal{P}(Y) \to [0, \infty]$ be a monotone function with $\rho(\emptyset) = 0$. Then it is well known that $\mathcal{M}(\rho) := \{M \subset Y : \forall A \in \mathcal{P}(Y)(\rho(A) = \rho(A \cap M) + \rho(A \setminus M))\}$ is an algebra and $\rho$ is an additive function on $\mathcal{M}(\rho)$. We apply this result in the following to the set functions $\rho = \underline{i}$ or $\rho = \overline{i}$, where $\underline{i}(A) := \underline{i}(1_A)$, $\overline{i}(A) := \overline{i}(1_A)$ for $A \in \mathcal{P}(Y)$. Furthermore

$$\mathcal{M}_0 := \{A \in \mathcal{P}(Y) : 1_A \in \mathcal{L}(i)\} = \{A \in \mathcal{P}(Y) : \underline{i}(A) = \overline{i}(A) \in \mathbb{R}\}$$

is called the system of *Loeb-integrable sets.* For $M_0 \in \mathcal{M}_0$ we also write $i^L(M_0)$ instead of $\underline{i}(M_0)(= \overline{i}(M_0))$. The systems $\mathcal{M}(\underline{i})$, respectively $\mathcal{M}(\overline{i})$, are called the system of *lower Loeb-measurable* sets, respectively *upper Loeb-measurable* sets.

## 2. The main results

The following theorem shows that our system $\mathcal{L}(i)$ is the same system as introduced by Loeb in [10], [11] and [4].

**1. Theorem.** *For a function $f : Y \to \mathbb{R}$ the following are equivalent:*
(i) $f \in \mathcal{L}(i)$;
(ii) *there exist $e_n \in \mathcal{E}^{fin}$ with* $\overline{i}(|f - e_n|) \underset{n\to\infty}{\to} 0$;
(iii) *there exists $e \in \mathcal{E}^{fin}$ with* $\overline{i}(|f - e|) = 0$.
*If $e_n$ is a sequence fulfilling* (ii)*, then* $i^L(f) = \lim_{n\to\infty} {}^\circ i(e_n)$*; if $e$ is a function fulfilling* (iii)*, then* $i^L(f) = {}^\circ i(e)$.

Let us remark that even for the formulation of Theorem 1 it is essential that we have defined the upper Loeb-integral $\overline{i}$ for functions with values in $^*\mathbb{R}$ and not only for functions with values in $\mathbb{R}$.

Most of the parts of the following theorem are due to Loeb (see [10], [4]). The proofs given here are different and use the properties of the inner and outer Loeb-integral.

**2. Theorem.** (i) $i^L : \mathcal{L}(i) \to \mathbb{R}$ *is a positive linear functional on the Stonian lattice* $\mathcal{L}(i)$.

(ii) $\mathcal{L}(i) \ni f_n \uparrow f \in \mathbb{R}^Y$ *and* $\sup_{n\in\mathbb{N}} i^L(f_n) < \infty$ *imply* $f \in \mathcal{L}(i)$ *and* $i^L(f_n) \uparrow i^L(f)$.

(iii) $f, h \in \mathcal{L}(i)$ *and* $f \leq g \leq h$ *with* $i^L(f) = i^L(h)$ *imply* $g \in \mathcal{L}(i)$.

The following theorem shows that the Loeb-integral $i^L$ on $\mathcal{L}(i)$ is in fact a classical Lebesgue-integral with respect to the measure space $(Y, \mathcal{M}(\overline{i}), \overline{i})$.

**3. Theorem.** (i) $\overline{i}$ *is a complete and saturated measure on the* $\sigma$*-algebra* $\mathcal{M}(\overline{i})$.

(ii) $\mathcal{M}(\overline{i}) = \{A \subset Y : (\forall M_0 \in \mathcal{M}_0)(A \cap M_0 \in \mathcal{M}_0)\}$.

(iii) $\mathcal{M}_0$ *is a* $\delta$*-ring and* $\mathcal{M}_0 = \{A \in \mathcal{M}(\overline{i}) : \overline{i}(A) \in \mathbb{R}\}$.

(iv) $\overline{i}(f) = \int f\, d\overline{i}$ *for each* $\mathcal{M}(\overline{i})$*-measurable function* $f : Y \to [0, \infty[$.

(v) $\mathcal{L}(i) = \{f \in \mathbb{R}^Y : f$ *is* $\overline{i}|\mathcal{M}(\overline{i})$*-Lebesgue-integrable*$\}$ *and* $i^L(f) = \int f\, d\overline{i}$ *for each* $f \in \mathcal{L}(i)$.

(vi) $0 \leq f \in \mathcal{L}(i)$ *and* $r > 0 \Rightarrow \{f > r\} \in \mathcal{M}_0$.

In the following theorem we consider the lower Loeb-integral and compare it with the classical Lebesgue-integral with respect to the measure space $(Y, \mathcal{M}(\underline{i}), \underline{i})$. For many problems in stochastics it turns out that the $\sigma$-algebra $\mathcal{M}(\underline{i})$ — which is in general strictly smaller than the $\sigma$-algebra $\mathcal{M}(\overline{i})$ — is the suitable $\sigma$-algebra. The reason is that many important functions are even $\mathcal{M}(\underline{i})$-measurable and not only $\mathcal{M}(\overline{i})$-measurable. Hence we obtain more information about those functions. Examples for such functions are $^\circ e$ for $e \in \mathcal{E}$ or the standard part map of regular topological spaces. In all those spaces the measure $\underline{i}$ (and not $\overline{i}$) allows us to construct Radon-measures of great interest.

**4. Theorem.** (i) $\underline{i}$ *is a complete measure on the* $\sigma$*-algebra* $\mathcal{M}(\underline{i})$.

(ii) $\mathcal{M}(\underline{i}) \subset \mathcal{M}(\overline{i})$.

(iii) $\mathcal{M}_0 \subset \{A \in \mathcal{M}(\underline{i}) : \underline{i}(A) \in \mathbb{R}\}$.

(iv) $\underline{i}(f) = \int f\, d\underline{i}$ *for each* $\mathcal{M}(\underline{i})$*-measurable function* $f : Y \to [0, \infty[$.

(v) $\mathcal{L}(i) \subset \{f \in \mathbb{R}^Y : f$ *is* $\underline{i}|\mathcal{M}(\underline{i})$*-Lebesgue-integrable*$\}$ *and* $i^L(f) = \int f\, d\underline{i}$ *for each* $f \in \mathcal{L}(i)$.

Let us remark (see Example 20 in §3) that it can happen that:

(i) $\underline{i}|\mathcal{M}(\underline{i})$ is not a saturated measure;

(ii) $\mathcal{M}(\underline{i}) \underset{\neq}{\subset} \mathcal{M}(\overline{i})$;

(iii) $\mathcal{M}_0 \underset{\neq}{\subset} \{A \in \mathcal{M}(\underline{i}) : \underline{i}(A) \in \mathbb{R}\}$;

(iv) $\mathcal{L}(i) \underset{\neq}{\subset} \{f \in \mathbb{R}^Y : f$ is $\underline{i}|\mathcal{M}(\underline{i})$-Lebesgue-integrable$\}$.

If $f : Y \to [0, \infty]$ is $\mathcal{M}(\underline{i})$-, respectively $\mathcal{M}(\overline{i})$-, measurable, then we write $\underline{i}(f)$, respectively $\overline{i}(f)$, for $\int f\, d\underline{i}$, respectively $\int f\, d\overline{i}$. This notation is justified by Theorem 3 (iv), respectively Theorem 4 (iv), for real-valued functions. Observe that for a function $f : Y \to [0, \infty]$ which attains the value $\infty$, $\underline{i}(f)$ and $\overline{i}(f)$ were not defined before.

The following example shows that internal contents lead in a natural way to upper and lower Loeb-integrals which coincide on sets with the inner and outer Loeb-measures.

If $\nu : \mathcal{R} \to {}^*[0,\infty[$ is an internal content on a ring $\mathcal{R} \subset \mathcal{P}(Y)$, then

$$\begin{aligned}\underline{\nu}(A) &:= \sup\{{}^\circ\nu(R) : \mathcal{R} \ni R \subset A\},\\ \overline{\nu}(A) &:= \inf\{{}^\circ\nu(R) : A \subset R \in \mathcal{R}\}\end{aligned}$$

denote the usual inner and outer Loeb-measure of $\nu$.

**5. Example.** Let $\nu : \mathcal{R} \to {}^*[0,\infty[$ be an internal content on a ring $\mathcal{R} \subset P(Y)$. Let $\mathcal{E}$ be the system of all internal sums $\sum_{j=1}^h \alpha_j 1_{R_j}$ with $\alpha_j \in {}^*\mathbb{R}$, $R_j \in \mathcal{R}$, $h \in {}^*\mathbb{N}$ and put

$$i_\nu(\sum_{j=1}^h \alpha_j 1_{R_j}) = \sum_{j=1}^h \alpha_j \nu(R_j).$$

Then $\mathcal{E}$ is an internal Stonian lattice, $i_\nu : \mathcal{E} \to {}^*\mathbb{R}$ is an internal positive linear functional and we have

(i) $\underline{\nu}(R) = \overline{\nu}(R) = {}^\circ\nu(R)$ for all $R \in \mathcal{R}$;
(ii) $\underline{\nu}(A) = \underline{i_\nu}(A)$, $\overline{\nu}(A) = \overline{i_\nu}(A)$ for all $A \subset Y$;
(iii) $\mathcal{M}_0 = \{A \subset Y : \underline{\nu}(A) = \overline{\nu}(A) < \infty\}$;
(iv) $\mathcal{R} \subset \mathcal{M}(\underline{\nu}) \subset \mathcal{M}(\overline{\nu})$;
(v) $\underline{\nu}$ is a complete measure on the $\sigma$-algebra $\mathcal{M}(\underline{\nu})$; $\overline{\nu}$ is a complete and saturated measure on the $\sigma$-algebra $\mathcal{M}(\overline{\nu})$.

The following theorem generalizes the $\widehat{S}$-continuity of the inner and outer Loeb-measure for finite internal contents (see [5]) to $\widehat{S}$-continuity of inner and outer Loeb-integrals.

**6. Theorem.** *Let $\mathcal{E}_1 \subset \mathcal{E}$ with cardinality of $\mathcal{E}_1$ smaller than or equal to $\widehat{S}$. Then*
(i) $\sup_{e\in\mathcal{E}_1} {}^\circ e$ *is $\mathcal{M}(\underline{i})$-measurable.*
*If $\mathcal{E}_1$ is furthermore an upwards directed system of nonnegative functions, then*
(ii) $\sup_{e\in\mathcal{E}_1} \overline{i}({}^\circ e) = \overline{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e)$;
(iii) $\sup_{e\in\mathcal{E}_1} \underline{i}({}^\circ e) = \underline{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e)$;
(iv) $\underline{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e) = \overline{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e)$.

The following two theorems are fundamental for the application of inner and outer Loeb-integrals to obtain similar representation results as were given for finite Loeb-measures in [5]; in these cases $Y_0$ is the set $ns^*X$ of near-standard points, which is in general not measurable. Applications of the following two theorems will be given in a forthcoming paper.

**7. Theorem.** *Let $\emptyset \neq Y_0 \subset Y$. Then*
(i) $\underline{i}|\mathcal{M}(\underline{i}) \cap Y_0$, $\overline{i}|\mathcal{M}(\overline{i}) \cap Y_0$ *are measures;*
(ii) $\underline{i}(f\, 1_{Y_0}) = \int f|_{Y_0} d\underline{i}|_{Y_0}$;
(iii) $\overline{i}(f\, 1_{Y_0}) = \int f|_{Y_0} d\overline{i}|_{Y_0}$;
*where we assume in* (ii)*, respectively* (iii)*, that the function $f : Y \to [0,\infty[$ is $\mathcal{M}(\underline{i})$-, respectively $\mathcal{M}(\overline{i})$-, measurable and $\underline{i}|_{Y_0}$, respectively $\overline{i}|_{Y_0}$, is the measure $\underline{i}|\mathcal{M}(\underline{i}) \cap Y_0$, respectively $\overline{i}|\mathcal{M}(\overline{i}) \cap Y_0$.*

If $f : Y \to [0,\infty]$ is $\mathcal{M}(\underline{i})$-, respectively $\mathcal{M}(\overline{i})$-, measurable, then we write $\underline{i}(f1_{Y_0})$, respectively $\overline{i}(f\, 1_{Y_0})$, for $\int f|_{Y_0} d\underline{i}|_{Y_0}$, respectively $\int f|_{Y_0} d\overline{i}|_{Y_0}$. This notation is justified by Theorem 7.

**8. Theorem.** *Let $\mathcal{E}_1 \subset \mathcal{E}$ be an upwards directed system of nonnegative functions with cardinality smaller than or equal to $\widehat{S}$. Then for each $Y_0 \subset Y$ we have*
(i) $\underline{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e 1_{Y_0}) = \sup_{e\in\mathcal{E}_1} \underline{i}({}^\circ e 1_{Y_0})$;
(ii) $\overline{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e 1_{Y_0}) = \sup_{e\in\mathcal{E}_1} \overline{i}({}^\circ e 1_{Y_0})$;
*where we assume in* (i) *and* (ii) *that the left-hand terms are finite.*

We show in Example 19 in §3 that the finiteness condition in Theorem 8 (ii) cannot be omitted.

## 3. Auxiliary lemmata and proof of the results

*Proof of Theorem* 1. (iii)⇒(ii) is trivial. (ii)⇒(i) follows by definition of $\overline{i}$ and $\mathcal{L}(i)$.
(i)⇒ (iii) By (i) there exist for each $n \in \mathbb{N}$ functions $g_n, h_n \in \mathcal{E}^{fin}$ with

$$(1) \qquad {}^\circ i(h_n) - 1/n \le i^L(f) \le {}^\circ i(g_n) + 1/n,$$

$$(2) \qquad g_n \le f \le h_n.$$

Put $\mathcal{H}_n := \{e \in \mathcal{E} : g_n \le e \le h_n\}$. Then $\bigcap_{i=1}^n \mathcal{H}_i \ne \emptyset$ by (2). Hence there exists by saturation $e \in \bigcap_{i=1}^\infty \mathcal{H}_i$. This $e$ fulfills (iii). The remaining assertions are obvious. □

**1. Lemma.** *We have for $g, g_1, g_2 \in {}^*\mathbb{R}^Y$ and $0 \le \alpha \in \mathbb{R}$*
(i) $\underline{i}(g) \le \overline{i}(g)$, $\overline{i}(-g) = -\underline{i}(g)$;
(ii) $\overline{i}(\alpha g) = \alpha\overline{i}(g)$, $\underline{i}(\alpha g) = \alpha\underline{i}(g)$;
(iii) $g_1 \le g_2 \Rightarrow \overline{i}(g_1) \le \overline{i}(g_2)$, $\underline{i}(g_1) \le \underline{i}(g_2)$;
(iv) $\overline{i}(g_1 + g_2) \le \overline{i}(g_1) + \overline{i}(g_2)$, $\underline{i}(g_1 + g_2) \ge \underline{i}(g_1) + \underline{i}(g_2)$, *whenever the right side is defined;*
(v) $\underline{i}(g_1 + g_2) \le \underline{i}(g_1) + \overline{i}(g_2)$ *if* $0 \le g_1, g_2$;
(vi) $i^L(g_1 + g_2) = \underline{i}(g_1) + \overline{i}(g_2)$, *if* $0 \le g_1, g_2$ *and* $g_1 + g_2 \in \mathcal{L}(i)$;
(vii) $0 \le f_n \uparrow f$, $f_n$, $f \in \mathbb{R}^Y \Rightarrow \overline{i}(f_n) \uparrow \overline{i}(f)$.

*Proof.* (i)–(v) follow directly from the definitions of $\underline{i}$ and $\overline{i}$. (vi) follows by using (v), (iv) and (i).
(vii) According to (iii) there exists $\alpha \ge 0$ with $\overline{i}(f_n) \uparrow \alpha$. Let w.l.o.g. $\alpha < \infty$ and let $\varepsilon \in \mathbb{R}_+$ be given. It suffices to show for each fixed $\delta \in \mathbb{R}_+$ that

$$(1) \qquad \overline{i}(f) \le (1 + \delta)(\alpha + \varepsilon).$$

According to the definition of $\overline{i}$ there exist $g_n \in \mathcal{E}^{fin}$ with $f_n \le g_n$ and ${}^\circ i(g_n) \le \overline{i}(f_n) + \varepsilon/2^n$. Then $h_n := g_1 \vee \ldots \vee g_n$ fulfill (use induction)

$$(2) \qquad f_n \le h_n \in \mathcal{E}^{fin}, {}^\circ i(h_n) \le \overline{i}(f_n) + \sum_{i=1}^n \varepsilon/2^i.$$

Put $\mathcal{H}_n := \{h \in \mathcal{E} : h_n \le h,\ i(h) \le \alpha + \varepsilon\}$. Then $\mathcal{H}_n$ are internal with $\emptyset \ne \mathcal{H}_n \downarrow$. Hence there exists $h \in \bigcap_{n=1}^\infty \mathcal{H}_n$. Then by (2) and $f_n \uparrow f$ we obtain $f \le (1 + \delta)h$. Hence we obtain (1). □

*Proof of Theorem* 2. (i) follows directly from Lemma 1 and Theorem 1.
(ii) As $0 \le f_n - f_1 \uparrow f - f_1 \in \mathbb{R}^Y$ we obtain the assertion from Lemma 1(vii) using (i).
(iii) follows from the definition of $\mathcal{L}(i)$. □

**2. Lemma.** *Let $e \in \mathcal{E}^{fin}$ with $e(Y) \subset fin({}^*\mathbb{R})$ be given. Then ${}^\circ e \in \mathcal{L}(i)$. If furthermore $0 \le e$, then $i^L({}^\circ e) \le {}^\circ i(e)$.*

*Proof.* Let w.l.o.g. $e \ge 0$ and put $e_n := (e - 1/n)^+$. Then

$$(1) \qquad 0 \le e_n \in \mathcal{E}^{fin},\ {}^\circ e_n \uparrow {}^\circ e;$$

$$(2) \qquad |e_n - {}^\circ e_n| \le \alpha e \text{ for all } 0 < \alpha \in \mathbb{R}.$$

By (2) we obtain $\overline{i}(|e_n - {}^\circ e_n|) = 0$. Hence ${}^\circ e_n \in \mathcal{L}(i)$ by Theorem 1. Since $\sup_{n\in\mathbb{N}} i^L({}^\circ e_n) \le {}^\circ i(e) < \infty$ we obtain the assertions using (1) and Theorem 2(ii). □

**3. Lemma.** *Let $f_0 : Y \to [0, \infty[$. Then*

$$\overline{i}(f_0) = \inf\{\lim i^L(f_n) : f_0 \le \lim f_n,\ \mathcal{L}(i) \ni f_n \uparrow\} =: \widetilde{i^L}(f_0).$$

*Proof.* If $\mathcal{L}(i) \ni f_n \uparrow$ and $f_0 \le \lim f_n$, then by Lemma 1 (vii):

$$\overline{i}(f_0) = \overline{i}(\lim_{n\in\mathbb{N}}(f_0 \wedge f_n)) = \lim_{n\in\mathbb{N}} \overline{i}(f_0 \wedge f_n) \le \lim_{n\in\mathbb{N}} \overline{i}(f_n).$$

Hence $\overline{i}(f_0) \le \widetilde{i^L}(f_0)$. To prove the converse let $\overline{i}(f_0) < \infty$. Let $e \in \mathcal{E}^{fin}$ with $f_0 \le e$. We have to prove

$$(1) \qquad \widetilde{i^L}(f_0) \le {}^\circ i(e).$$

Put $f_n := {}^\circ e \wedge n = {}^\circ(e \wedge n)$. As $e \wedge n \in \mathcal{E}^{fin}$ we obtain $f_n \in \mathcal{L}(i)$ and $i^L(f_n) \le {}^\circ i(e \wedge n) \le {}^\circ i(e)$ by Lemma 2. Hence $\widetilde{i^L}(f_0) \le \lim_{n\to\infty} i^L(f_n) \le {}^\circ i(e)$, as $\mathcal{L}(i) \ni f_n \uparrow$ and $f_0 \le \lim f_n$. □

The following concept is essentially due to Loeb (see [10], [4]).

**4. Definition.** We call $\langle J, j\rangle$ a complete integration structure if $J \subset \mathbb{R}^Y$ is a Stonian vector-lattice and $j : J \to \mathbb{R}$ is a positive linear functional with the following properties:

$$(1) \qquad J \ni f_n \uparrow f \in \mathbb{R}^Y \text{ and } \sup_{n\in\mathbb{N}} j(f_n) < \infty \Rightarrow f \in J \text{ and } j(f_n) \uparrow j(f),$$

$$(2) \qquad f, h \in J, \quad f \le g \le h \text{ and } j(f) = j(h) \Rightarrow g \in J.$$

According to Theorem 2, $\langle \mathcal{L}(i), i^L\rangle$ is a complete integration structure. For a complete integration structure put, for $g \in \mathbb{R}^Y$,

- $\widetilde{j}(g) := \inf\{\lim j(f_n) : g \le \lim f_n, J \ni f_n \uparrow\}$,
- $\underset{\sim}{j}(g) := \sup\{\lim j(f_n) : \lim f_n \le g, J \ni f_n \downarrow\}$,
- $\mathcal{M}_0 := \{A \subset Y : 1_A \in J\}$,
- $\mathcal{M} := \{A \subset Y : A \cap M_0 \in \mathcal{M}_0 \text{ for all } M_0 \in \mathcal{M}_0\}$,
- $\widetilde{j}(A) := \widetilde{j}(1_A)$.

By Theorem 7.1 on page 103 of Floret [3] and by Proposition 10.7 and Corollaries 12.20 and 12.22 of Pfeffer [12] one obtains the following classical result.

**5. Lemma.** *Let $\langle J, j\rangle$ be a complete integration structure. Then $J = \{f \in \mathbb{R}^Y : \underset{\sim}{j}(f) = \widetilde{j}(f) \in \mathbb{R}\}$ and*

(i) *$\widetilde{j}$ is a complete and saturated measure on the $\sigma$-algebra $\mathcal{M}(\widetilde{j})$;*
(ii) *$\mathcal{M}(\widetilde{j}) = \{A \subset Y : (\forall M_0 \in \mathcal{M}_0)\ A \cap M_0 \in \mathcal{M}_0\}$;*

(iii) $\mathcal{M}_0$ *is a* $\delta$*-ring and* $\mathcal{M}_0 = \{A \in \mathcal{M}(\widetilde{j}) : \widetilde{j}(A) \in \mathbb{R}\}$;
(iv) $\widetilde{j}(f) = \int f\, d\widetilde{j}$ *for each* $\mathcal{M}(\widetilde{j})$*-measurable function* $f : Y \to [0, \infty[$;
(v) $J = \{f \in \mathbb{R}^Y : f$ *is* $\widetilde{j}|\mathcal{M}(\widetilde{j})$*-Lebesgue-integrable*$\}$ *and* $j(f) = \int f\, d\widetilde{j}$ *for all* $f \in J$;
(vi) $0 \le f \in J$ *and* $0 < r \Rightarrow \{f > r\} \in \mathcal{M}_0$.

*Proof of Theorem* 3. $\langle J, j\rangle := \langle \mathcal{L}(i), i^L\rangle$ is a complete integration according to Theorem 2. Hence $\mathcal{L}(i) = \{f \in \mathbb{R}^Y : \underset{\sim}{i}^L(f) = \widetilde{i^L}(f) \in \mathbb{R}\}$ by Lemma 5. By Lemma 3 we have $\widetilde{j}(f) = \widetilde{i^L}(f) = \overline{i}(f)$ for $f : Y \to [0, \infty[$, and hence $\widetilde{j}(A) = \widetilde{i^L}(A) = \overline{i}(A)$. Therefore we obtain (i)–(vi) of Theorem 3 from (i)–(vi) of Lemma 5. □

**6. Lemma.** *Let* $e \in \mathcal{E}$ *with* $|e| \le f \in \mathbb{R}^Y$ *and* $\underline{i}(f) < \infty$ *be given. Then* ${}^\circ e \in \mathcal{L}(i)$ *and* $i^L({}^\circ e) = {}^\circ i(e)$.

*Proof.* As $\underline{i}(|e|) \le \underline{i}(f) < \infty$, it is easily seen that $i(|e|)$ is finite (otherwise $i(\frac{n|e|}{i(|e|)}) = n$ and $\frac{n|e|}{i(|e|)} \le f$). Hence ${}^\circ e \in \mathcal{L}(i)$ by Lemma 2. Now let w.l.o.g. $e \ge 0$. As $i^L({}^\circ e) \le {}^\circ i(e)$ by Lemma 2, it remains to prove ${}^\circ i(e) \le i^L({}^\circ e)$. To this aim let $\varepsilon \in \mathbb{R}_+$ be given. As $|e| \le f \in \mathbb{R}^Y$ we obtain $|e - {}^\circ e| \le \varepsilon f$. Hence $\underline{i}(|e - {}^\circ e|) \le \varepsilon \underline{i}(f)$, whence $\underline{i}(|e - {}^\circ e|) = 0$. The assertion follows now, using Lemma 1 (v), from

$$ {}^\circ i(e) = \underline{i}(e) \le \underline{i}(|e - {}^\circ e| + {}^\circ e) \le \underline{i}(|e - {}^\circ e|) + \overline{i}({}^\circ e) = \overline{i}({}^\circ e) = i^L({}^\circ e). $$

□

**7. Lemma.** *For* $Y_0 \subset Y$ *we have*
(i) $\overline{i}(Y_0) < \infty \Rightarrow i^L(M_0) = \overline{i}(Y_0)$ *for some* $\mathcal{M}_0 \ni M_0 \supset Y_0$;
(ii) $\underline{i}(Y_0) < \infty \Rightarrow i^L(M_0) = \underline{i}(Y_0)$ *for some* $\mathcal{M}_0 \ni M_0 \subset Y_0$;
(iii) $M_0 \in \mathcal{M}_0 \Rightarrow i^L(M_0) = \overline{i}(M_0 \cap Y_0) + \underline{i}(M_0 \setminus Y_0)$.

*Proof.* (i) By definition of $\overline{i}(Y_0)$ there exist

$$ e_n \in \mathcal{E}^{fin},\ 1_{Y_0} \le e_n \le 1,\ e_n \downarrow \text{ with } \overline{i}(Y_0) = \lim_{n\to\infty} {}^\circ i(e_n). \tag{1} $$

According to Lemma 2 and Theorem 2 we have ${}^\circ e_n \in \mathcal{L}(i)$ and

$$ \overline{i}(Y_0) \ge \lim_{n\to\infty} i^L({}^\circ e_n) = i^L(\lim_{n\to\infty} {}^\circ e_n). \tag{2} $$

Put $M_0 := \{\lim_{n\to\infty} {}^\circ e_n = 1\}$. Then $Y_0 \subset M_0$ according to (1) and $\overline{i}(Y_0) \underset{(2)}{\ge} \overline{i}(\lim {}^\circ e_n) \ge \overline{i}(M_0)$, and hence $\overline{i}(Y_0) = \overline{i}(M_0)$. As $f := \lim_{n\to\infty} {}^\circ e_n \in \mathcal{L}(i)$ and $0 \le f \le 1$ we have $\{f = 1\} = \bigcap_{n=2}^\infty \{f > 1 - 1/n\} \in \mathcal{M}_0$ by Theorem 3 (vi) and (iii).

(ii) According to the definition of $\underline{i}(Y_0)$ there exist $\mathcal{E} \ni e_n \uparrow$ with $0 \le e_n \le 1_{Y_0}$ and

$$ \infty > \underline{i}(Y_0) = \lim_{n\to\infty} {}^\circ i(e_n) = \lim_{n\to\infty} i^L({}^\circ e_n) = i^L(\lim_{n\to\infty} {}^\circ e_n), $$

where the last two equalities follow from Lemma 6 and Theorem 2. Put $f_0 := \lim_{n\to\infty} {}^\circ e_n$ and $M_0 := \{f_0 > 0\}$. Then $M_0 \subset Y_0$ and $f_0 \in \mathcal{L}(i)$. According to Theorem 3 (vi) we obtain that $M_n := \{f_0 > 1/n\} \in \mathcal{M}_0$. As $M_n \uparrow M_0$ and $\sup i^L(1_{M_n}) = \sup \underline{i}(M_n) \le \underline{i}(Y_0) < \infty$, we obtain $M_0 \in \mathcal{M}_0$ by Theorem 2 (ii). The assertion follows now from

$$ \underline{i}(Y_0) = i^L(f_0) = \overline{i}(f_0) \le \overline{i}(M_0) = i^L(M_0). $$

(iii) Using Lemma 1 we obtain the assertion by

$$\begin{aligned} i^L(M_0) &= \underline{i}(1_{M_0}) \le \underline{i}(1_{M_0\setminus Y_0}) + \overline{i}(1_{M_0\cap Y_0}) = \underline{i}(M_0 \setminus Y_0) + \overline{i}(1_{M_0} \setminus 1_{M_0\setminus Y_0}) \\ &\le \underline{i}(M_0 \setminus Y_0) + \overline{i}(-1_{M_0\setminus Y_0}) + \overline{i}(M_0) \\ &= \underline{i}(M_0 \setminus Y_0) + i^L(M_0) - \underline{i}(M_0 \setminus Y_0) \\ &= i^L(M_0). \end{aligned}$$

□

**8. Lemma.** *Let $Y \supset Y_n \downarrow Y_0$ and $\underline{i}(Y_1) < \infty$. Then $\underline{i}(Y_n) \downarrow \underline{i}(Y_0)$.*

*Proof.* Put $\alpha := \lim_{n\to\infty} \underline{i}(Y_n)$. Let $\varepsilon \in \mathbb{R}_+$. We have to show

$$\alpha - \varepsilon \le \underline{i}(Y_0). \tag{1}$$

By definition of $\underline{i}$ there exist $0 \le e_n \in \mathcal{E}^{fin}$ with $e_n \le 1_{Y_n}$ and $\underline{i}(Y_n) \le {}^\circ i(e_n)+\varepsilon/2^n$. Let $g_n := e_1 \wedge \ldots \wedge e_n$. Then

$$0 \le g_n \in \mathcal{E}^{fin}, \quad g_n \le 1_{Y_n}, \quad g_n \downarrow , \tag{2}$$

$$\underline{i}(Y_n) \le {}^\circ i(g_n) + \sum_{i=1}^{n} \varepsilon/2^i \text{ (which follows by induction).} \tag{3}$$

Put $\mathcal{C}_n := \{e \in \mathcal{E} : 0 \le e \le g_n \text{ and } i(e) \ge \alpha - \varepsilon\}$. Then $\emptyset \ne \mathcal{C}_n \downarrow$ . By saturation there exists $e \in \bigcap_{n=1}^\infty \mathcal{C}_n$. Hence $0 \le e \in \mathcal{E}^{fin}$, $i(e) \ge \alpha - \varepsilon$ and $e \le g_n$. Therefore $e \le 1_{Y_0}$ and hence (1) follows. □

**9. Lemma.** *Let $\mathcal{C}$ be a system of internal sets with $\underline{i}(C) = \infty$ for all $C \in \mathcal{C}$. Assume that $\mathcal{C}$ is downwards directed, has the finite intersection property and fulfills $|\mathcal{C}| \le |\widehat{S}|$. Then $\underline{i}(\bigcap_{C\in\mathcal{C}} C) = \infty$.*

*Proof.* Put $\mathcal{S}_C^n := \{e \in \mathcal{E} : 0 \le e \le 1_C,\ i(e) \ge n\}$. Then $\{\mathcal{S}_C^n : C \in \mathcal{C}, n \in \mathbb{N}\}$ is a system of internal sets with finite intersection and cardinality $\le |\widehat{S}|$. Hence $\bigcap_{n\in\mathbb{N},C\in\mathcal{C}} \mathcal{S}_C^n \ne \emptyset$. Let $e \in \bigcap_{n\in\mathbb{N},C\in\mathcal{C}} \mathcal{S}_C^n$. Then $e \in \mathcal{E}$ and $e \le 1_{\bigcap_{C\in\mathcal{C}} C}$ with $i(e) \ge n$ for all $n \in \mathbb{N}$. Hence for each $n \in \mathbb{N}$ there exists $e \in \mathcal{E}^{fin}$ with $i(|e|) \ge n$ and $e \le 1_{\bigcap_{C\in\mathcal{C}} C}$. Hence $\underline{i}(\bigcap_{C\in\mathcal{C}} C) = \infty$. □

**10. Lemma.** *Let $Y_0 \subset Y$ with $\underline{i}(Y_0) = \infty$ and $c_0 := \sup\{\underline{i}(T) : T \subset Y_0,\ \underline{i}(T) < \infty\} < \infty$. Then there exists an internal set $C \subset Y_0$ with $\underline{i}(C) = \infty$.*

*Proof.* As $\underline{i}(Y_0) = \infty$, there exist $0 \le e_n \in \mathcal{E}^{fin}$ with $e_n \le 1_{Y_0}$ and ${}^\circ i(e_n) \underset{n\to\infty}{\to} \infty$. Choose $n_0$ with $c_0 < {}^\circ i(e_{n_0})$ and put $C := \{e_{n_0} > 0\}$. Then $C$ is an internal subset of $Y_0$ and $c_0 < \underline{i}(e_{n_0}) \le \underline{i}(C)$. Hence $\underline{i}(C) = \infty$ by definition of $c_0$. □

**11. Lemma.** *Let $0 \le f \in \mathbb{R}^Y$ and $\underline{i}(\{f \ne 0\})$. Then $\underline{i}(f) = 0$.*

*Proof.* Let $0 \le e \le f$ with $e \in \mathcal{E}^{fin}$ be given. Then as $e$ is internal there exists $c \in \mathbb{R}_+$ with $e \le c$. Hence $e \le c1_{\{f\ne0\}}$ and therefore ${}^\circ i(e) = \underline{i}(e) \le \underline{i}(c1_{\{f\ne0\}}) = 0$. □

**12. Lemma.** *Let $B \in \mathcal{M}(\overline{i})$. Then $B \in \mathcal{M}(\underline{i})$ if the following condition is fulfilled:*

$$A \text{ internal, } \underline{i}(A) = \infty \text{ and } \sup\{\underline{i}(T) : T \subset A,\ \underline{i}(T) < \infty\} < \infty$$
$$\Rightarrow \underline{i}(A \cap B) = \infty \text{ or } \underline{i}(A \setminus B) = \infty.$$

*Proof.* We have to show that for $A \subset Y$

$$\underline{i}(A) \le \underline{i}(A \cap B) + \underline{i}(A \setminus B). \tag{1}$$

If $\underline{i}(A) < \infty$, then by Lemma 7 (ii) there exists $M_0 \in \mathcal{M}_0$ with $M_0 \subset A$ and $\underline{i}(A) = i^L(M_0)$. Then $M_0 \cap B$, $M_0 \cap \overline{B} \in \mathcal{M}_0$ by Theorem 3 (ii) and hence

$$\underline{i}(A) = i^L(M_0) = i^L(M_0 \cap B) + i^L(M_0 \setminus B) \le \underline{i}(A \cap B) + \underline{i}(A \setminus B).$$

If $\infty = \underline{i}(A) = \sup\{\underline{i}(T) : T \subset A,\ \underline{i}(T) < \infty\}$, then we obtain as just proved

$$\underline{i}(T) \le \underline{i}(T \cap B) + \underline{i}(T \setminus B)$$

for all $T$ with $\underline{i}(T) < \infty$. Hence (1) follows. Now consider the remaining case, namely $\underline{i}(A) = \infty$ and $\sup\{\underline{i}(T) : T \subset A,\ \underline{i}(T) < \infty\} < \infty$. Then by Lemma 10 there exists an internal set $A'$ with $A' \subset A$ and $\underline{i}(A') = \infty$. Hence we obtain $\underline{i}(A' \cap B) = \infty$ or $\underline{i}(A' \setminus B) = \infty$ by assumption. As $A' \subset A$, this implies (1). □

Let us remark that Lemma 12 yields a condition for $\mathcal{M}(\underline{i}) = \mathcal{M}(\bar{i})$.

*Proof of Theorem* 4. (ii) It suffices to show $A \notin \mathcal{M}(\bar{i}) \Rightarrow A \notin \mathcal{M}(\underline{i})$.

As $A \notin \mathcal{M}(\bar{i})$, we obtain by Theorem 3 (ii) that there exists $M_0 \in \mathcal{M}_0$ with $A \cap M_0 \notin \mathcal{M}_0$. Now $\bar{i}(A \cap M_0) \le \bar{i}(M_0) < \infty$ and hence $\underline{i}(A \cap M_0) < \bar{i}(A \cap M_0)$ by definition of $\mathcal{M}_0$. Using Lemma 7 (iii) we obtain

$$\underline{i}(M_0) = \bar{i}(M_0 \cap A) + \underline{i}(M_0 \setminus A) > \underline{i}(M_0 \cap A) + \underline{i}(M_0 \setminus A);$$

hence $A \notin \mathcal{M}(\underline{i})$.

(iii) Let $A \in \mathcal{M}_0$. We have to prove (see Lemma 1 (iv)) that

$$\underline{i}(B) \le \underline{i}(B \cap A) + \underline{i}(B \setminus A) \text{ for each } B \subset Y. \tag{1}$$

If $\underline{i}(B) < \infty$, then by Lemma 7 (ii) there exists $M_0 \in \mathcal{M}_0$ with $M_0 \subset B$ and $i^L(M_0) = \underline{i}(B)$. Hence

$$\underline{i}(B) = i^L(M_0) = i^L(M_0 \cap A) + i^L(M_0 \setminus A) \le \underline{i}(B \cap A) + \underline{i}(B \setminus A),$$

i.e. (1) holds. Let $\underline{i}(B) = \infty$. According to Lemma 1 (v) we obtain $\underline{i}(B) \le \bar{i}(B \cap A) + \underline{i}(B \setminus A) \le i^L(A) + \underline{i}(B \setminus A)$; hence $\underline{i}(B \setminus A) = \infty$ and (1) holds.

(i) As $\mathcal{M}(\underline{i})$ is an algebra, to prove that $\mathcal{M}(\underline{i})$ is a $\sigma$-algebra, it is sufficient to show

$$A_n \in \mathcal{M}(\underline{i}) \wedge A_n \downarrow A \Rightarrow A \in \mathcal{M}(\underline{i}). \tag{2}$$

As $A \in \mathcal{M}(\bar{i})$ by (ii), it suffices to prove according to Lemma 12

$$\begin{gathered} B \text{ internal, } \underline{i}(B) = \infty \text{ and } \sup\{\underline{i}(T) : T \subset B,\ \underline{i}(T) < \infty\} < \infty \\ \Rightarrow \underline{i}(B \cap A) = \infty \text{ or } \underline{i}(B \setminus A) = \infty. \end{gathered} \tag{3}$$

Ad (3): Let w.l.o.g. $\underline{i}(B \setminus A) < \infty$. We inductively construct internal sets $C_n$ with

$$C_n \subset B \cap A_n,\ C_n \downarrow \text{ and } \underline{i}(C_n) = \infty. \tag{4}$$

Then $\underline{i}(\bigcap_{n=1}^{\infty} C_n) = \infty$ by Lemma 9 and hence $\underline{i}(B \cap A) \ge \underline{i}(\bigcap_{n=1}^{\infty} C_n) = \infty$, whence (3) holds.

As $\underline{i}(B \setminus A_n) \le \underline{i}(B \setminus A) < \infty$, we obtain from $A_n \in \mathcal{M}(\underline{i})$

$$\underline{i}(B \cap A_n) = \infty \text{ for } n \in \mathbb{N}. \tag{5}$$

According to Lemma 10 applied to $Y_0 := B \cap A_1$, there exists an internal set $C_1 \subset B \cap A_1$ with $\underline{i}(C_1) = \infty$. Assume inductively that $C_1, \ldots, C_n$ are constructed with $C_1 \supset \ldots \supset C_n, C_i \subset B \cap A_i$ and $\underline{i}(C_i) = \infty$ for $i = 1, \ldots, n$. As $A_{n+1} \in \mathcal{M}(\underline{i})$, we obtain

$$\infty = \underline{i}(C_n) = \underline{i}(C_n \cap A_{n+1}) + \underline{i}(C_n \setminus A_{n+1}),$$

which implies $\underline{i}(C_n \cap A_{n+1}) = \infty$ as $\underline{i}(C_n \setminus A_{n+1}) \le \underline{i}(B \setminus A_{n+1}) \le \underline{i}(B \setminus A) < \infty$.

According to Lemma 10 applied to $Y_0 := C_n \cap A_{n+1}$ we obtain an internal set $C_{n+1} \subset C_n \cap A_{n+1}$ with $\underline{i}(C_{n+1}) = \infty$. Hence (4) is proven by induction.

Now we prove that $\underline{i}$ is a measure on $\mathcal{M}(\underline{i})$. As $\underline{i}$ is additive on $\mathcal{M}(\underline{i})$, it suffices to show that

$$\mathcal{M}(\underline{i}) \ni A_n \uparrow A \Rightarrow \underline{i}(A_n) \to \underline{i}(A). \tag{6}$$

If $\underline{i}(A) < \infty$, then $\underline{i}(A \setminus A_n) \to 0$ by Lemma 8, and hence (6) holds.

Now let $\underline{i}(A) = \infty$ and assume indirectly that $\sup \underline{i}(A_n) =: c_1 < \infty$. First, let $\sup\{\underline{i}(T) : T \subset A, \underline{i}(T) < \infty\} = \infty$. According to Lemma 7 (ii) there exists $M_0 \subset A$ with $i^L(M_0) > c_1$. As $A_n \in \mathcal{M}(\bar{i})$ by (ii), we obtain $M_0 \cap A_n \in \mathcal{M}_0$ by Theorem 3 (ii). Hence we obtain the following contradiction:

$$c_1 < i^L(M_0) = \lim_{n\to\infty} i^L(M_0 \cap A_n) = \lim_{n\to\infty} \underline{i}(M_0 \cap A_n) \le \lim \underline{i}(A_n) \le c_1.$$

Now let $\sup\{\underline{i}(T) : T \subset A, \underline{i}(T) < \infty\} < \infty$. We construct internal sets $C_n \subset A \setminus A_n$ with $C_n \downarrow$ and $\underline{i}(C_n) = \infty$. This yields a contradiction since $\bigcap_{n=1}^{\infty} C_n \subset \bigcap_{n=1}^{\infty} (A \setminus A_n) = \emptyset$ implies $C_n = \emptyset$ for some $n$.

As $\underline{i}(A \setminus A_1) = \infty$, we obtain by Lemma 10 an internal set $C_1 \subset A \setminus A_1$ with $\underline{i}(C_1) = \infty$. Now assume inductively that internal sets $C_n \subset C_{n-1} \subset \ldots \subset C_1$ with $C_j \subset A \setminus A_j$ and $\underline{i}(C_j) = \infty$ for $j = 1, \ldots, n$ are constructed. As $A \setminus A_{n+1} \in \mathcal{M}(\underline{i})$, we have

$$\infty = \underline{i}(C_n) = \underline{i}(C_n \cap (A \setminus A_{n+1})) + \underline{i}(C_n \setminus (A \setminus A_{n+1})).$$

Since $\underline{i}(C_n \setminus (A \setminus A_{n+1})) \le \underline{i}(A_{n+1}) < \infty$, we obtain $\underline{i}(C_n \cap (A \setminus A_{n+1})) = \infty$. Hence by Lemma 10 there exists an internal set $C_{n+1} \subset C_n \cap (A \setminus A_{n+1})$ with $\underline{i}(C_{n+1}) = \infty$. Hence we have proven that $\underline{i}$ is a measure on $\mathcal{M}(\underline{i})$. The completeness of $\underline{i}|\mathcal{M}(\underline{i})$ follows from the monotonicity of $\underline{i}$ and the definition of $\mathcal{M}(\underline{i})$.

(v) Now let $0 \le f \in \mathcal{L}(i)$. Then according to Theorem 3 (vi) and (iii) we have for each $r > 0$

$$\{f > r\} \in \mathcal{M}_0 \subset \mathcal{M}(\underline{i}). \tag{7}$$

Hence $f$ is $\mathcal{M}(\underline{i})$-measurable. It suffices to prove $i^L(f) = \int f \, d\underline{i}$. According to (7) there exist $f_n = \sum_{j=1}^{k_n} \alpha_{j,n} 1_{M_{j,n}}$ with $M_{j,n} \in \mathcal{M}_0$ and $f_n \uparrow f$. Hence according to Theorem 2 and the properties of the Lebesgue-integral

$$i^L(f) = \lim i^L(f_n) = \lim \sum_{j=1}^{k_n} \alpha_{j,n} \underline{i}(M_{j,n}) = \lim \int f_n \, d\underline{i} = \int f \, d\underline{i}.$$

To prove (iv) we show at first that for each $\underline{i}|\mathcal{M}(\underline{i})$-Lebesgue-integrable function $f \ge 0$ there exist $0 \le f_0 \in \mathcal{L}(i)$ with

$$f_0 = f \ \underline{i}|\mathcal{M}(\underline{i})\text{-a.e. and } i^L(f_0) = \int f \, d\underline{i}. \tag{8}$$

Choose $\mathcal{M}(\underline{i})$-elementary functions $f_n$ with $0 \le f_n \uparrow f$. Using Lemma 7 (ii) one can construct $g_n \in \mathcal{L}(i)$ with $0 \le g_n \le f_n$, $g_n \uparrow$ and $g_n = f_n$ $\underline{i}|\mathcal{M}(\underline{i})$-a.e. Put $f_0 := \lim g_n \ge 0$. Using (v) and Theorem 2 it is easy to see that $0 \le f_0 \in \mathcal{L}(i)$ fulfills (8).

(iv) Let $0 \leq f$ be $\mathcal{M}(\underline{i})$-measurable. We first prove

$$\underline{i}(f) \geq \int f \, d\underline{i}. \tag{9}$$

As $0 \leq f$ is $\mathcal{M}(\underline{i})$-measurable, there exist $\mathcal{M}(\underline{i})$-elementary functions

$$f_n = \sum_{j=1}^{k_n} \alpha_{j,n} 1_{M_{j,n}} \uparrow f.$$

Hence by Lemma 1 we obtain

$$\underline{i}(f) \geq \underline{i}(f_n) \geq \sum_{j=1}^{k_n} \underline{i}(\alpha_{j,n} 1_{M_{j,n}}) = \sum_{j=1}^{k_n} \alpha_{j,n} \underline{i}(M_{j,n}) = \int f_n \, d\underline{i} \uparrow \int f \, d\underline{i},$$

which shows (9). Because of (9) it suffices to show

$$\underline{i}(f) \leq \int f \, d\underline{i} \text{ for } \int f \, d\underline{i} < \infty. \tag{10}$$

According to (8) there exists $0 \leq f_0 \in \mathcal{L}(i)$ with

$$f_0 = f \ \underline{i}|\mathcal{M}(\underline{i})\text{-a.e. and } i^L(f_0) = \int f \, d\underline{i}. \tag{11}$$

To prove (10) it therefore suffices to show

$$\underline{i}(f) \leq i^L(f_0). \tag{12}$$

We obtain from Lemma 1 (v)

$$\underline{i}(f) \leq \underline{i}(|f - f_0|) + \overline{i}(f_0) = \underline{i}(|f - f_0|) + i^L(f_0).$$

As $\underline{i}(|f - f_0|) = 0$ by Lemma 11, we obtain (12). □

*Proof of Example* 5. (i) follows from the monotonicity of $\nu$ on $\mathcal{R}$ and the definition of $\underline{\nu}$ and $\overline{\nu}$.

(ii) We first show $\underline{\nu}(A) \leq \underline{i_\nu}(A)$. To prove this consider the following two cases:

(1) $\nu(R)$ is finite for all $R \in \mathcal{R}$ with $R \subset A$;

(2) there exists $R_0 \in \mathcal{R}$ with $R_0 \subset A$ and $\nu(R_0)$ infinite.

Ad (1): Let $R \in \mathcal{R}, R \subset A$ be given. Then $\mathcal{E}^{fin} \ni e := 1_R \leq 1_A$ and hence ${}^\circ\nu(R) = {}^\circ i_\nu(e) \leq \underline{i_\nu}(1_A)$. Whence $\underline{\nu}(A) \leq \underline{i_\nu}(1_A) = \underline{i_\nu}(A)$.

Ad (2): Let $\alpha \in \mathbb{R}_+$ and put $e := \frac{\alpha}{\nu(R_0)} 1_{R_0}$. Then $e \in \mathcal{E}^{fin}, i_\nu(e) = \alpha$ and $e \leq 1_A$. Hence $\underline{i_\nu}(1_A) \geq {}^\circ i_\nu(e) = \alpha$. As $\alpha \in \mathbb{R}_+$ is arbitrary, we obtain $\underline{i_\nu}(A) = \infty \geq \underline{\nu}(A)$.
It remains to prove $\underline{i_\nu}(A) \leq \underline{\nu}(A)$. Let $e \in \mathcal{E}^{fin}$ with $0 \leq e \leq 1_A$ be given. We have $e = \sum_{j=1}^n \alpha_j 1_{R_j}$ with disjoint $\emptyset \neq R_j \in \mathcal{R}$. Then $R := \bigcup_{\alpha_j > 0} R_j \in \mathcal{R}$ and $R \subset A$. Furthermore $e \leq 1_R$ and hence

$${}^\circ i_\nu(e) \leq {}^\circ i_\nu(1_R) \underset{(i)}{=} \underline{\nu}(R) \leq \underline{\nu}(A).$$

Hence $\underline{i_\nu}(A) = \underline{i_\nu}(1_A) \leq \underline{\nu}(A)$. This proves $\underline{\nu}(A) = \underline{i_\nu}(A)$.

The proof for $\overline{\nu}(A) = \overline{i_\nu}(A)$ runs similarly.

(iii) As $\mathcal{M}_0 = \{A \subset Y : \underline{i_\nu}(1_A) = \overline{i}_\nu(1_A) < \infty\}$ by definition, we obtain (iii) by (ii).

(iv) As $\underline{\nu}(A) = \underline{i_\nu}(A)$, $\overline{\nu}(A) = \overline{i_\nu}(A)$ for each $A \subset Y$, we obtain $\mathcal{M}(\underline{\nu}) \subset \mathcal{M}(\overline{\nu})$ by Theorem 4 (ii). Let $R \in \mathcal{R}$; it remains to show $R \in \mathcal{M}(\underline{\nu})$. Let $A \subset Y$. As $\nu$ is a content on $\mathcal{R}$ we obtain from the definition of $\underline{\nu}$ that $\underline{\nu}(A) \le \underline{\nu}(A \cap R) + \underline{\nu}(A \setminus R)$. Hence $R \in \mathcal{M}(\underline{\nu})$.

(v) follows from Theorem 4 and Theorem 3 using (ii). □

**13. Lemma.** *Let* $0 \le e \in \mathcal{E}$ *and let* $0 \le r \in \mathbb{R}$. *Then there exists* $e_r \in \mathcal{E}$ *with* $1_{\{^\circ e > r\}} \le e_r \le 1_{\{^\circ e \ge r\}}$.

*Proof.* Put $e_r = 1 \wedge n(e - e \wedge r)$ with $n \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then $e_r \in \mathcal{E}$ and $e_r$ fulfills the above inequalities. □

**14. Lemma.** *Let* $B \subset Y$ *be internal with* $\underline{i}(B) = \infty$. *Then for every* $r \in [0,1]$ *and* $e \in \mathcal{E}$ *with* $0 \le e$ *we have*

$$\underline{i}(B \cap \{^\circ e \le r\}) = \infty \textit{ or } \underline{i}(B \cap \{^\circ e \ge r\}) = \infty.$$

*Proof.* As $\underline{i}(B) = \infty$, there exists $g_n \in \mathcal{E}^{fin}$ with

$$0 \le g_n \le 1_B, \quad {}^\circ i(g_n) \uparrow \infty. \tag{1}$$

By Lemma 13 there exists $e_r \in \mathcal{E}$ with

$$1_{\{^\circ e > r\}} \le e_r \le 1_{\{^\circ e \ge r\}}. \tag{2}$$

From (1) and (2) we obtain

$$g_n \wedge e_r \in \mathcal{E}^{fin}, g_n \wedge e_r \le 1_{B \cap \{^\circ e \ge r\}}, \tag{3}$$

$$\mathcal{E}^{fin} \ni g_n - g_n \wedge e_r \le 1_{B \cap \{^\circ e \le r\}}. \tag{4}$$

As ${}^\circ i(g_n) = {}^\circ i(g_n \wedge e_r) + {}^\circ i(g_n - g_n \wedge e_r)$ and ${}^\circ i(g_n) \uparrow \infty$, there exists a subsequence with ${}^\circ i(g_n \wedge e_r) \to \infty$ or ${}^\circ i(g_n - g_n \wedge e_r) \to \infty$. Whence $\underline{i}(B \cap \{^\circ e \le r\}) = \infty$ or $\underline{i}(B \cap \{^\circ e \ge r\}) = \infty$. □

**15. Lemma.** (i) $e \in \mathcal{E}^{fin} \Rightarrow {}^\circ e$ *is* $\sigma(\mathcal{M}_0)$*-measurable.*

(ii) $e \in \mathcal{E} \Rightarrow {}^\circ e$ *is* $\mathcal{M}(\underline{i})$*-measurable.*

*Proof.* Using that $\mathcal{E}$ is a Stonian lattice we may assume in (i) and (ii) that $0 \le e \le 1$.

(i) According to Lemma 2 we have ${}^\circ e \in \mathcal{L}(i)$. The assertion follows now from Theorem 3 (vi).

(ii) We first show that ${}^\circ e$ is $\mathcal{M}(\overline{i})$-measurable. Let $M_0 \in \mathcal{M}_0$ be given. We prove

$$1_{M_0} \wedge {}^\circ e \in \mathcal{L}(i). \tag{1}$$

Let $\varepsilon \in \mathbb{R}_+$ be given. There exists $g$ with

$$g \in \mathcal{E}^{fin},\ 1_{M_0} \le g \le 1 \text{ and } i^L({}^\circ g) \le {}^\circ i(g) \le i^L(M_0) + \varepsilon. \tag{2}$$

As ${}^\circ(e \wedge g) \le 1_{M_0} \wedge {}^\circ e + {}^\circ g - 1_{M_0}$, we obtain from Lemma 1 (v)

$$\underline{i}({}^\circ(e \wedge g)) \le \underline{i}(1_{M_0} \wedge {}^\circ e) + \overline{i}({}^\circ g - 1_{M_0}).$$

As ${}^\circ(e \wedge g), {}^\circ g \in \mathcal{L}(i)$ by Lemma (2), we obtain from (2)

$$\overline{i}(1_{M_0} \wedge {}^\circ e) \le \overline{i}({}^\circ(e \wedge g)) \le \underline{i}(1_{M_0} \wedge {}^\circ e) + \varepsilon.$$

This proves (1). Now let $r > 0$. Then by Theorem 3 (vi) we have $\{^\circ e > r\} \cap M_0 = \{1_{M_0} \wedge {}^\circ e > r\} \underset{(1)}{\in} \mathcal{M}_0$. Hence we obtain by Theorem 3 (ii) that

$${}^\circ e \text{ is } \mathcal{M}(\overline{i})\text{-measurable}. \tag{3}$$

Now let $0 \leq r \in \mathbb{R}$ be given and put $B_r := \{^\circ e \geq r\}$. Then $B_r \in \mathcal{M}(\bar{i})$ according to (3). To prove $B_r \in \mathcal{M}(\underline{i})$ we have to show according to Lemma 12:

$$(4) \qquad \begin{cases} A \text{ internal}, \quad \underline{i}(A) = \infty \text{ and } \sup\{\underline{i}(T) : T \subset A, \underline{i}(T) < \infty\} < \infty \\ \qquad\qquad \Rightarrow \underline{i}(A \cap B_r) = \infty \text{ or } \underline{i}(A \setminus B_r) = \infty. \end{cases}$$

Define $r_0 := \sup\{s \in [0,1] : \underline{i}(A \cap B_s) = \infty\} \in [0,1]$. If $r < r_0$, then $\underline{i}(A \cap B_r) = \infty$. If $r > r_0$, then $\underline{i}(A \cap B_r) < \infty$ and hence using Lemma 14: $\underline{i}(A \setminus B_r) = \underline{i}(A \cap \{^\circ e < r\}) = \infty$. This proves (3) for $r \neq r_0$. It remains to prove (3) if $r = r_0$. Let $T_n := A \cap \{e > r_0 - 1/n\}$. Then $T_n$ are internal with $T_n \downarrow$ and $\underline{i}(T_n) = \infty$ by definition of $r_0$. Hence $\underline{i}(\bigcap_{n=1}^\infty T_n) = \infty$ by Lemma 9. As $\bigcap_{n=1}^\infty T_n \subset A \cap \{^\circ e \geq r_0\}$, we obtain $\underline{i}(A \cap B_{r_0}) = \infty$. □

**16. Lemma.** *If* $0 \leq e \in \mathcal{E}$, *then*
(i) $\underline{i}(e) = \bar{i}(e) = {}^\circ i(e)$;
(ii) $\underline{i}(^\circ e) = \bar{i}(^\circ e)$.

*Proof.* (i) If $e \in \mathcal{E}^{fin}$, then the assertion follows by definition of $\bar{i}$ and $\underline{i}$. If $e \notin \mathcal{E}^{fin}$, then $i(e)$ is positive infinite. It suffices to prove $\underline{i}(e) = \infty$. Let $\alpha \in \mathbb{R}_+$ and put $e_\alpha = \frac{\alpha}{i(e)} e$. Then $i(e_\alpha) = \alpha, e_\alpha \in \mathcal{E}^{fin}$ and $e_\alpha \leq e$. Hence $\underline{i}(e) \geq \alpha$ for all $\alpha \in \mathbb{R}_+$.

(ii) $^\circ e$ is $\mathcal{M}(\underline{i})$-measurable by Lemma 15. As

$$\begin{aligned} \bar{i}(^\circ e) &= \int {}^\circ e \, d\bar{i} = \lim_{n\to\infty} \int (^\circ e \wedge n) d\bar{i} = \lim_{n\to\infty} \bar{i}(^\circ e \wedge n), \\ \underline{i}(^\circ e) &= \int {}^\circ e \, d\underline{i} = \lim_{n\to\infty} \int (^\circ e \wedge n) d\underline{i} = \lim_{n\to\infty} \underline{i}(^\circ e \wedge n), \end{aligned}$$

we may assume that $0 \leq e \leq 1$. According to Theorem 4 (iv) it suffices to show that $^\circ e \in \mathcal{L}(i)$ if $^\circ e$ is $\underline{i}|\mathcal{M}(\underline{i})$-Lebesgue-integrable. Therefore let $^\circ e$ be $\underline{i}|\mathcal{M}(\underline{i})$-Lebesgue-integrable. We show $\{^\circ e > r\} \in \mathcal{M}_0$ for each $r > 0$; by Lemma 13 there exists $0 \leq e_r \in \mathcal{E}$ with $1_{\{^\circ e > r\}} \leq e_r \leq 1_{\{^\circ e \geq r\}}$. Hence by (i)

$$\bar{i}(\{^\circ e > r\}) \leq \bar{i}(e_r) = \underline{i}(e_r) \leq \underline{i}\{^\circ e \geq r\} < \infty.$$

As $\{^\circ e > r\} \in \mathcal{M}(\bar{i})$ by Lemma 15 (ii), we obtain that $\{^\circ e > r\} \in \mathcal{M}_0$ by Theorem 3 (iii). As $^\circ e$ is $\mathcal{M}(\bar{i})$-measurable, we obtain therefore for $k, n \in \mathbb{N}$ that $\{k/2^n \leq {}^\circ e < (k+1)/2^n\} \in \mathcal{M}_0$ (use Theorem 3 (iii)). Hence $f_n := \sum_{k=1}^{n2^n} \frac{k}{2^n} 1_{\{\frac{k}{2^n} \leq {}^\circ e < \frac{k+1}{2^n}\}} \in \mathcal{L}(i)$ and $f_n \uparrow {}^\circ e$. By Theorem 4 (v) we obtain

$$\sup i^L(f_n) = \sup \int f_n \, d\underline{i} \leq \int {}^\circ e \, d\underline{i} < \infty.$$

Hence Theorem 2 (ii) implies $^\circ e \in \mathcal{L}(i)$ □

*Proof of Theorem* 6. We first prove (ii)–(iv) under the assumption that

$$(1) \qquad 0 \leq e \leq 1 \text{ for all } e \in \mathcal{E}_1.$$

(ii) Assume that w.l.o.g.

$$(2) \qquad \alpha := \sup_{e \in \mathcal{E}_1} \bar{i}(^\circ e) < \infty.$$

Let $\varepsilon \in \mathbb{R}_+$ be given. Define for $e \in \mathcal{E}_1, n \in \mathbb{N}$

$$\mathcal{H}_e^n := \{g \in \mathcal{E} : e1_{\{e \geq 1/n\}} \leq g \leq 2, \, i(g) \leq \alpha + \varepsilon\}.$$

Then $\mathcal{H}_e^n$ are internal sets and we show that

$$(3) \qquad \bigcap_{i=1}^k \mathcal{H}_{e_i}^{n_i} \neq \emptyset \text{ for all } e_i \in \mathcal{E}_1, n_i \in \mathbb{N} \text{ and } k \in \mathbb{N}.$$

To this aim let $n := n_1 \vee \ldots \vee n_k$. As $\mathcal{E}_1$ is upwards directed, there exists $\widehat{e} \in \mathcal{E}_1$ with $e_1 \vee \ldots \vee e_k \leq \widehat{e}$. Hence

$$(4) \qquad e_1 1_{\{e_1 \geq 1/n_1\}} \vee \ldots \vee e_k 1_{\{e_k \geq 1/n_k\}} \leq \widehat{e}\, 1_{\{\widehat{e} \geq 1/n\}}.$$

As $\widehat{e} \in \mathcal{E}_1$, we have $\bar{i}(^\circ\widehat{e}) \leq \alpha$ by (2). Hence there exist $1 \geq g_1 \in \mathcal{E}^{fin}$ and $1 \geq \delta \in \mathbb{R}_+$ such that

$$(5) \qquad {}^\circ\widehat{e} \leq g_1 \text{ with } (1+\delta) i(g_1) \leq \alpha + \varepsilon.$$

Then

$$(6) \qquad \widehat{e} 1_{\{\widehat{e} \geq 1/n\}} \leq (1+\delta) g_1 =: g \leq 2.$$

Then $2 \geq g \in \mathcal{E}$ and $i(g) \leq \alpha + \varepsilon$ according to (5) and hence $g \in \bigcap_{i=1}^k \mathcal{H}_{e_i}^{n_i}$ by (4) and (6). This proves (3). By compactness there exists $\tilde{e} \in \bigcap_{e \in \mathcal{E}_1, n \in \mathbb{N}} \mathcal{H}_e^n$. Then by Lemma 2

$$(7) \qquad i^L(^\circ\tilde{e}) \leq {}^\circ i(\tilde{e}) \leq \alpha + \varepsilon,$$

$$(8) \qquad e 1_{\{e \geq 1/n\}} \leq \tilde{e} \text{ for all } e \in \mathcal{E}_1,\ n \in \mathbb{N}.$$

This implies $^\circ e \leq {}^\circ\tilde{e}$ for all $e \in \mathcal{E}_1$. Hence $\sup\limits_{e \in \mathcal{E}_1} {}^\circ e \leq {}^\circ\tilde{e}$ and (7) implies

$$\bar{i}(\sup_{e \in \mathcal{E}_1} {}^\circ e) \leq \bar{i}(^\circ\tilde{e}) = i^L(^\circ\tilde{e}) \leq \alpha + \varepsilon.$$

As $\varepsilon \in \mathbb{R}_+$ was arbitrary, we obtain (ii) under assumption (1).

(iii) follows using Lemma 16 (ii) from

$$\sup_{e \in \mathcal{E}_1} \underline{i}(^\circ e) \leq \underline{i}(\sup_{e \in \mathcal{E}_1} {}^\circ e) \leq \bar{i}(\sup_{e \in \mathcal{E}_1} {}^\circ e) \underset{\text{(ii)}}{=} \sup_{e \in \mathcal{E}_1} \bar{i}(^\circ e) = \sup_{e \in \mathcal{E}_1} \underline{i}(^\circ e).$$

(iv) follows from (ii) and (iii) using Lemma 16 (ii).

Now we prove (i). Here we may assume w.l.o.g. that $\mathcal{E}_1$ is upwards directed and (1) holds. Put $f := \sup\limits_{e \in \mathcal{E}_1} {}^\circ e$. We first show that $f$ is $\mathcal{M}(\bar{i})$-measurable. To this aim choose $M_0 \in \mathcal{M}_0$. We show that

$$(9) \qquad 1_{M_0} \wedge f \in \mathcal{L}(i).$$

Then for $0 < r$ we have according to Theorem 3 (vi)

$$\{f > r\} \cap M_0 = \{1_{M_0} \wedge f > r\} \in \mathcal{M}_0.$$

Hence $f$ is $\mathcal{M}(\bar{i})$-measurable according to Theorem 3 (ii).

To prove (9) let $\varepsilon \in \mathbb{R}_+$ be given. Choose $h$ with

$$(10) \qquad h \in \mathcal{E}^{fin}, 1_{M_0} \leq h \leq 1, \quad i^L(^\circ h) \leq {}^\circ i(h) \leq i^L(M_0) + \varepsilon.$$

Then for $e \in \mathcal{E}_1$

$$(11) \qquad {}^\circ(e \wedge h) \leq 1_{M_0} \wedge {}^\circ e + {}^\circ h - 1_{M_0}.$$

Hence according to (iv) and Lemma 1 (v) we have

$$\begin{aligned} \bar{i}(1_{M_0} \wedge f) &\leq \bar{i}(\sup_{e \in \mathcal{E}_1} {}^\circ(e \wedge h)) \underset{\text{(iv)}}{=} \underline{i}(\sup_{e \in \mathcal{E}_1} {}^\circ(e \wedge h)) \underset{(11)}{\leq} \underline{i}(1_{M_0} \wedge f) + \bar{i}(^\circ h - 1_{M_0}) \\ &\underset{(10)}{\leq} \underline{i}(1_{M_0} \wedge f) + \varepsilon. \end{aligned}$$

This proves (9). Now we prove that

$$(12) \qquad f \text{ is } \mathcal{M}(\underline{i})\text{-measurable.}$$

Let $r > 0$. Then $\{f > r\} \in \mathcal{M}(\bar{i})$ as just proven. To prove $\{f > r\} \in \mathcal{M}(\underline{i})$ it remains according to Lemma 12 to show:

$$\text{(13)} \quad A \text{ internal } \wedge\ \underline{i}(A) = \infty \ \Rightarrow\ \underline{i}(A \cap \{f > r\}) = \infty \text{ or } \underline{i}(A \cap \{f \le r\}) = \infty.$$

Assume $\underline{i}(A \cap \{f > r\}) < \infty$. Then $\underline{i}(A \cap \{^\circ e > r\}) < \infty$ for all $e \in \mathcal{E}_1$. As $^\circ e$ is $\mathcal{M}(\underline{i})$-measurable according to Lemma 15 (ii), we have

$$\text{(14)} \qquad \underline{i}(A \cap \{^\circ e \le r\}) = \infty.$$

As $A \cap \{^\circ e \le r\} \subset A \cap \{e \le r + 1/n\}$ for each $n \in \mathbb{N}$, we obtain from (14)

$$\text{(15)} \qquad \underline{i}(A \cap \{e \le r + 1/n\}) = \infty.$$

As $\mathcal{C} := \{A \cap \{e \le r + 1/n\} : e \in \mathcal{E}_1, n \in \mathbb{N}\}$ is a system of internal sets with the finite intersection property and $|\mathcal{C}| \le |\widehat{S}|$, we have

$$\text{(16)} \qquad \underline{i}(A \cap \bigcap_{e \in \mathcal{E}_1, n \in \mathbb{N}} \{e \le r + 1/n\}) = \infty$$

by Lemma 9. Now $\underline{i}(A \cap \{f \le r\}) = \infty$ follows from $A \cap \bigcap_{e \in \mathcal{E}_1, n \in \mathbb{N}} \{e \le r + 1/n\} = A \cap \{f \le r\}$ and (16), i.e. (13) is proven.

It remains to prove (ii)–(iv) without the assumption (1). It is clear that (ii)–(iv) follows under the assumption

$$0 \le e \le n \text{ for all } e \in \mathcal{E}_1.$$

Hence (ii) follows using (i) from

$$\begin{aligned} \sup_{g \in \mathcal{E}_1} \bar{i}(^\circ g) &= \sup_{g \in \mathcal{E}_1} \int {^\circ g}\, d\bar{i} = \sup_{n \in \mathbb{N}} \sup_{g \in \mathcal{E}_1} \int {^\circ (g \wedge n)}\, d\bar{i} = \sup_{n \in \mathbb{N}} \sup_{g \in \mathcal{E}_1} \bar{i}(^\circ g \wedge n) \\ &= \sup_{n \in \mathbb{N}} \int (\sup_{g \in \mathcal{E}_1} {^\circ g}) \wedge n\, d\bar{i} = \int \sup_{g \in \mathcal{E}_1} {^\circ g}\, d\bar{i} = \bar{i}(\sup_{g \in \mathcal{E}_1} {^\circ g}). \end{aligned}$$

(iii) follows similary. (iv) follows from (ii) and (iii) using Lemma 16 (ii). □

**17. Lemma.** *Let $f_1, f_2 : Y \to [0, \infty[$ be $\mathcal{M}(\bar{i})$-measurable and $Y_0 \subset Y$. Then*

$$\bar{i}(f_1 1_{Y_0} + f_2 1_{Y_0}) = \bar{i}(f_1 1_{Y_0}) + \bar{i}(f_2 1_{Y_0}).$$

*Proof.* According to Lemma 1 (vii) we may assume that $f_1, f_2$ are bounded. According to Lemma 1 (iv) it suffices to prove

$$\text{(1)} \qquad \bar{i}(f_1 1_{Y_0} + f_2 1_{Y_0}) \ge \bar{i}(f_1 1_{Y_0}) + \bar{i}(f_2 1_{Y_0});$$

assume w.l.o.g. that $\bar{i}(f_1 1_{Y_0} + f_2 1_{Y_0}) < \infty$. To prove (1) let $e \in \mathcal{E}^{fin}$ with

$$\text{(2)} \qquad e \ge f_1 1_{Y_0} + f_2 1_{Y_0},\ e \text{ bounded},$$

be chosen. According to Lemma 2, $^\circ e \in \mathcal{L}(i)$ and hence $f_1 \wedge {^\circ e}, {^\circ e} - f_1 \wedge {^\circ e}$ are $\mathcal{M}(\bar{i})$-measurable. Therefore $f_1 \wedge {^\circ e}, {^\circ e} - f_1 \wedge {^\circ e} \in \mathcal{L}(i)$ (use Theorem 3 (v)). Hence using (2)

$$i^L(^\circ e) = i^L(f_1 \wedge {^\circ e}) + i^L(^\circ e - f_1 \wedge {^\circ e}) \ge \bar{i}(f_1 1_{Y_0}) + \bar{i}(f_2 1_{Y_0}).$$

Since $i^L(^\circ e) \le {^\circ i}(e)$ by Lemma 2, we obtain (1). □

**18. Lemma.** *Let $f_n : Y \to [0, \infty[$ be $\mathcal{M}(\underline{i})$-measurable and $Y_0 \subset Y$. Then*

(i) $\underline{i}(f_1 1_{Y_0} + f_2 1_{Y_0}) < \infty \Rightarrow \underline{i}(f_1 1_{Y_0} + f_2 1_{Y_0}) = \underline{i}(f_1 1_{Y_0}) + \underline{i}(f_2 1_{Y_0})$;

(ii) $f_n \uparrow f \in \mathbb{R}^Y \wedge \underline{i}(f 1_{Y_0}) < \infty \Rightarrow \underline{i}(f_n 1_{Y_0}) \uparrow \underline{i}(f 1_{Y_0})$.

*Proof.* (i) According to Lemma 1 (iv) it suffices to prove

$$\underline{i}(f_1 1_{Y_0} + f_2 1_{Y_0}) \leq \underline{i}(f_1 1_{Y_0}) + \underline{i}(f_2 1_{Y_0}). \tag{1}$$

Let $e \in \mathcal{E}^{fin}$ with

$$0 \leq e \leq f_1 1_{Y_0} + f_2 1_{Y_0} \tag{2}$$

be given. Then we have

$$^\circ e \leq f_1 \wedge {}^\circ e + f_2 \wedge {}^\circ e, \quad f_1 \wedge {}^\circ e \leq f_1 1_{Y_0}, \quad f_2 \wedge {}^\circ e \leq f_2 1_{Y_0}.$$

Hence

$$i^L({}^\circ e) \leq i^L(f_1 \wedge {}^\circ e) + i^L(f_2 \wedge {}^\circ e) \leq \underline{i}(f_1 1_{Y_0}) + \underline{i}(f_2 1_{Y_0}).$$

As $\underline{i}(f_1 1_{Y_0} + f_2 1_{Y_0}) < \infty$, we obtain from (2) and Lemma 6 that $i^L({}^\circ e) = {}^\circ i(e)$. This implies (1).

(ii) We have to prove

$$\lim_{n\to\infty} \underline{i}(f_n 1_{Y_0}) \geq \underline{i}(f 1_{Y_0}). \tag{3}$$

Choose $e_n \in \mathcal{E}^{fin}$ with

$$e_n \leq e_{n+1},\ e_n \leq f 1_{Y_0},\ {}^\circ i(e_n) \uparrow \underline{i}(f 1_{Y_0}). \tag{4}$$

As $\underline{i}(f 1_{Y_0}) < \infty$, we obtain by Lemma 6 that ${}^\circ i(e_n) = i^L({}^\circ e_n)$ and hence

$$\begin{aligned}\underline{i}(f 1_{Y_0}) &= \lim_{n\to\infty} {}^\circ i(e_n) = \lim_{n\to\infty} i^L({}^\circ e_n) = i^L(\lim_{n\to\infty} {}^\circ e_n) \\ &= i^L(\lim_{n\to\infty} ({}^\circ e_n \wedge f_n)) = \lim_{n\to\infty} i^L({}^\circ e_n \wedge f_n) \leq \lim_{n\to\infty} \underline{i}(f_n 1_{Y_0}).\end{aligned}$$

□

*Proof of Theorem* 7. $\overline{i}|\mathcal{M}(\overline{i}) \cap Y_0$ is a measure according to Lemma 17 and Lemma 1 (vii). $\overline{i}(f 1_{Y_0}) = \int f_{|Y_0} d\overline{i}_{|Y_0}$ follows for $\mathcal{M}(\overline{i})$-elementary functions by Lemma 17 and Lemma 1 (ii). The general case follows then from Lemma 1 (vii).

It remains to prove

$$\underline{i}|\mathcal{M}(\underline{i}) \cap Y_0 \text{ is a measure,} \tag{1}$$

$$\underline{i}(f 1_{Y_0}) = \int f|_{Y_0}\, d\underline{i}|_{Y_0}. \tag{2}$$

Ad(1): Let $M_1 \cap Y_0$ and $M_2 \cap Y_0$ be disjoint with $M_1, M_2 \in \mathcal{M}(\underline{i})$. If $\underline{i}((M_1 \cap Y_0) \cup (M_2 \cap Y_0)) < \infty$, then

$$\underline{i}((M_1 \cap Y_0) \cup (M_2 \cap Y_0)) = \underline{i}(M_1 \cap Y_0) + \underline{i}(M_2 \cap Y_0)$$

by Lemma 18 (i). To prove that $\underline{i}|\mathcal{M}(\underline{i}) \cap Y_0$ is additive it suffices therefore to show

$$\underline{i}(M_1 \cap Y_0) < \infty,\ \underline{i}(M_2 \cap Y_0) < \infty \Rightarrow \underline{i}((M_1 \cap Y_0) \cup (M_2 \cap Y_0)) < \infty. \tag{3}$$

As $M_1 \in \mathcal{M}(\underline{i})$ we have with $A := Y_0 \cap (M_1 \cup M_2)$ :

$$\underline{i}(A) = \underline{i}(A \cap M_1) + \underline{i}(A \setminus M_1) \leq \underline{i}(Y_0 \cap M_1) + \underline{i}(Y_0 \cap M_2) < \infty,$$

i.e. (3) holds.

Now let $\mathcal{M}(\underline{i}) \ni M_n \uparrow M_\infty$ be given. To prove (1), it suffices to show

$$\underline{i}(M_n \cap Y_0) \uparrow \underline{i}(M_\infty \cap Y_0). \tag{4}$$

If $\underline{i}(M_\infty \cap Y_0) < \infty$, we obtain (4) from Lemma 18 (ii). Therefore, let $\underline{i}(M_\infty \cap Y_0) = \infty$ and assume indirectly that

$$\lim_{n\to\infty} \underline{i}(M_n \cap Y_0) \leq c_1 \text{ for some } c_1 \in \mathbb{R}. \tag{5}$$

We consider the following two cases:

$$\sup\{\underline{i}(T) : T \subset M_\infty \cap Y_0, \quad \underline{i}(T) < \infty\} = \infty, \tag{6}$$

$$\sup\{\underline{i}(T) : T \subset M_\infty \cap Y_0, \quad \underline{i}(T) < \infty\} < \infty. \tag{7}$$

Assume that (6) holds. Hence according to Lemma 7 (ii) we obtain

$$\sup\{\underline{i}(M_0) : M_0 \subset M_\infty \cap Y_0,\, M_0 \in \mathcal{M}_0\} = \infty.$$

Choose $M_0 \in \mathcal{M}_0$ with $M_0 \subset M_\infty \cap Y_0$ and $i^L(M_0) > c_1$. As $M_0 \cap M_n \in \mathcal{M}_0$ and $M_0 \cap M_n \uparrow M_0 \cap M_\infty = M_0$, we obtain the following contradiction:

$$\begin{aligned} c_1 < i^L(M_0) &= \lim_{n\to\infty} i^L(M_0 \cap M_n) = \lim_{n\to\infty} \underline{i}(M_0 \cap M_n) \\ &\leq \lim_{n\to\infty} \underline{i}(Y_0 \cap M_n) \leq c_1. \end{aligned}$$

Assume that (7) holds. We construct sets $C_n$ with

$$\begin{cases} C_n \text{ internal}, \quad C_n \subset M_\infty \cap Y_0 \setminus M_n \cap Y_0, \\ C_n \downarrow, \quad \underline{i}(C_n) = \infty. \end{cases} \tag{8}$$

Since $C_n$ by (8) are internal with $\bigcap_{n=1}^\infty C_n = \emptyset$, we obtain $C_n = \emptyset$ for some $n \in \mathbb{N}$, contradicting $\underline{i}(C_n) = \infty$. Thus it remains to construct $C_n$ fulfilling (8). As $\underline{i}(M_\infty \cap Y_0 \setminus M_1 \cap Y_0) = \infty$ we obtain an internal set $C_1 \subset M_\infty \cap Y_0 \setminus M_1 \cap Y_0$ with $\underline{i}(C_1) = \infty$ by Lemma 10. Now assume inductively that internal sets $C_1 \supset \ldots \supset C_n$ with $C_j \subset M_\infty \cap Y_0 \setminus M_j \cap Y_0$ and $\underline{i}(C_j) = \infty$ are constructed. As $M_\infty \setminus M_{n+1} \in \mathcal{M}(\underline{i})$ we have

$$\infty = \underline{i}(C_n) = \underline{i}(C_n \cap Y_0) = \underline{i}((C_n \cap Y_0) \cap (M_\infty \setminus M_{n+1})) + \underline{i}((C_n \cap Y_0) \setminus (M_\infty \setminus M_{n+1})).$$

Since

$$\underline{i}((C_n \cap Y_0) \setminus (M_\infty \setminus M_{n+1})) \leq \underline{i}(M_{n+1} \cap Y_0) < \infty,$$

we obtain $\underline{i}((C_n \cap Y_0) \cap (M_\infty \setminus M_{n+1})) = \infty$. Hence by Lemma 10 there exists an internal set $C_{n+1} \subset (C_n \cap Y_0) \cap (M_\infty \setminus M_{n+1})$ with $\underline{i}(C_{n+1}) = \infty$. This proves (8) for $n + 1$. Hence (1) is proven.

Ad(2): If $f = 1_M$ with $M \in \mathcal{M}(\underline{i})$, then (2) follows by definition of $\underline{i}|_{Y_0}$. Now let $0 \leq f$ be elementary. Then $f = \sum_{j=1}^n \alpha_j \, 1_{M_j}$ with disjoint $M_j \in \mathcal{M}(\underline{i})$ and $\alpha_j > 0$ given. If $\underline{i}(f 1_{Y_0}) < \infty$, then (2) follows from Lemma 18 (i) and Lemma 1 (ii). Now let $\underline{i}(f \; 1_{Y_0}) = \infty$ and $\alpha := \max_{j=1}^n \alpha_j$. Then $\underline{i}(\alpha 1_{\sum_{j=1}^n M_j \cap Y_0}) = \infty$ and hence $\underline{i}(1_{\sum_{j=1}^n M_j \cap Y_0}) = \infty$. Hence by (1) there exists $j_0$ with $\underline{i}(M_{j_0} \cap Y_0) = \infty$. Therefore $\infty = \sum_{j=1}^n \alpha_j \underline{i}(M_j \cap Y_0) = \int f|_{Y_0} d\underline{i}|_{Y_0}$. This proves (2) for $\mathcal{M}(\underline{i})$-elementary functions. Using Lemma 18 (ii) we obtain (2) for $\mathcal{M}(\underline{i})$-measurable functions $f \geq 0$ if $\underline{i}(f 1_{Y_0}) < \infty$.

Now consider the case $\underline{i}(f 1_{Y_0}) = \infty$. We first show

$$\underline{i}([f \wedge n] 1_{Y_0}) \uparrow \infty. \tag{9}$$

Assume indirectly that

$$c := \sup \underline{i}([f \wedge n] 1_{Y_0}) < \infty. \tag{10}$$

Let $e \leq f \; 1_{Y_0}$ with $e \in \mathcal{E}^{fin}$ given. Then there exists $n \in \mathbb{N}$ with $e \leq n$ as $e$ is internal. Hence

$$e \leq [f \wedge n] 1_{Y_0}. \tag{11}$$

From (11) we obtain that

$$^\circ i(e) \leq \underline{i}([f \wedge n]1_{Y_0}) \leq c.$$

Hence $\underline{i}(f\ 1_{Y_0}) \leq c$, contradicting our assumption $\underline{i}(f1_{Y_0}) = \infty$. Therefore (9) holds. Consequently it suffices to show (2) for $\mathcal{M}(\underline{i})$-measurable function $f$ with $0 \leq f \leq 1$ and $\underline{i}(f1_{Y_0}) = \infty$. Hence we have to show that $\int f_{|Y_0} d\underline{i}_{|Y_0} = \infty$. Let indirectly

$$c := \int f_{|Y_0} d\underline{i}_{|Y_0} < \infty. \tag{12}$$

Then

$$c = \sum_{n=1}^{\infty} \int f 1_{\{\frac{1}{n+1} < f \leq \frac{1}{n}\} \cap Y_0} d\underline{i}_{|Y_0}.$$

Now we have by (12)

$$\underline{i}(\{\frac{1}{n+1} < f \leq \frac{1}{n}\} \cap Y_0) \leq (n+1) \int f_{|Y_0} d\underline{i}_{|Y_0} < \infty.$$

Hence according to Lemma 7 (ii) there exists $M_n \in \mathcal{M}_0$ with $M_n \subset \{\frac{1}{n+1} < f \leq \frac{1}{n}\} \cap Y_0$ and $\underline{i}(M_n) = \underline{i}(\{\frac{1}{n+1} < f \leq \frac{1}{n}\} \cap Y_0) < \infty$. Therefore according to (1)

$$\underline{i}|_{Y_0}(\{\frac{1}{n+1} < f \leq \frac{1}{n}\} \cap Y_0 - M_n) = 0. \tag{13}$$

Put $M_\infty := \biguplus_{n=1}^{\infty} M_n$; then from (1) and (13) we obtain

$$\underline{i}_{|Y_0}(\{f > 0\} \cap Y_0 \setminus M_\infty) = 0. \tag{14}$$

Hence

$$\underline{i}(f1_{Y_0}1_{\{f>0\}} - f1_{M_\infty}) \leq 1 \cdot \underline{i}(Y_0 \cap \{f > 0\} - M_\infty) = 0. \tag{15}$$

From (15), Lemma 1 (v), Lemma 1 (vii) and (12) we obtain

$$\begin{aligned} \underline{i}(f1_{Y_0}) &= \underline{i}(f1_{Y_0} - f1_{M_\infty} + f1_{M_\infty}) \ \leq \ \underline{i}(f1_{Y_0} - f1_{M_\infty}) + \bar{i}(f1_{M_\infty}) \\ &= \lim_{n\to\infty} \bar{i}(f1_{\bigcup_{i=1}^n M_i}) \ = \ \lim_{n\to\infty} \underline{i}(f1_{\bigcup_{i=1}^n M_i}) \\ &= \lim_{n\to\infty} \int f1_{\bigcup_{i=1}^n M_i} d\underline{i}_{|Y_0} \leq \int f1_{Y_0} d\underline{i}_{|Y_0} < \infty. \end{aligned}$$

This contradicts our assumption $\underline{i}(f1_{Y_0}) = \infty$. □

*Proof of Theorem* 8. (i) It suffices to prove

$$\alpha := \underline{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e1_{Y_0}) \leq \sup_{e\in\mathcal{E}_1} \underline{i}({}^\circ e1_{Y_0}).$$

By Theorem 7 (ii) we may assume that $0 \leq e \leq 1$ for $e \in \mathcal{E}_1$. Now let $\varepsilon \in \mathbb{R}_+$ be given. As $\alpha$ is finite, there exists $g \in \mathcal{E}^{fin}$ with

$$g \leq (\sup_{e\in\mathcal{E}_1} {}^\circ e)1_{Y_0}, \tag{1}$$

$$i^L({}^\circ g) = {}^\circ i(g) \geq \alpha - \varepsilon. \tag{2}$$

Hence by (2), (1) and Theorem 6 (iii) we have

$$\begin{aligned} \alpha - \varepsilon \quad &\leq \quad \underline{i}({}^\circ g) \quad = \quad \underline{i}({}^\circ g \wedge \sup_{e\in\mathcal{E}_1} {}^\circ e) \quad = \quad \underline{i}(\sup_{e\in\mathcal{E}_1} {}^\circ(g \wedge e)) \\ &= \quad \sup_{e\in\mathcal{E}_1} \underline{i}({}^\circ(g \wedge e)) \quad \leq \quad \sup_{e\in\mathcal{E}_1} \underline{i}({}^\circ e1_{Y_0}). \end{aligned}$$

As $\varepsilon \in \mathbb{R}_+$ was arbitrary, we obtain the assertion.

(ii) It suffices to show

$$\beta := \bar{i}((\sup_{e\in\mathcal{E}_1} {}^\circ e)1_{Y_0}) \leq \sup_{e\in\mathcal{E}_1} \bar{i}({}^\circ e\, 1_{Y_0}).$$

According to Theorem 7 (iii) we may assume that $0 \leq e \leq 1$ for $e \in \mathcal{E}_1$.

As $\beta$ is finite, there exists $g$ with

$$g \in \mathcal{E}^{fin},\ \sup_{e\in\mathcal{E}_1} {}^\circ e \cdot 1_{Y_0} \leq g \leq 1.$$

Then

$$\sup_{e\in\mathcal{E}_1} {}^\circ e \wedge {}^\circ g = \sup_{e\in\mathcal{E}_1} {}^\circ(e \wedge g) \in \mathcal{L}(i)$$

according to Theorem 6 (iv). Then according to Theorem 6 there exists a sequence $e_n \in \mathcal{E}_1$ with $e_n \wedge g \leq e_{n+1} \wedge g$ and

$$\infty > \bar{i}(\sup_{e\in\mathcal{E}_1} {}^\circ(e\wedge g)) = \lim_{n\to\infty} \bar{i}({}^\circ(e_n \wedge g)) = \bar{i}(\sup_{n\in\mathbb{N}} {}^\circ(e_n \wedge g)).$$

Hence $\sup_{e\in\mathcal{E}_1} {}^\circ(e\wedge g) = \sup_{n\in\mathbb{N}} {}^\circ(e_n \wedge g)$ $\bar{i}$-a.e., and this implies according to Theorem 7 (iii)

$$\begin{aligned}\bar{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e\, 1_{Y_0}) &= \bar{i}(\sup_{e\in\mathcal{E}_1} {}^\circ(e\wedge g)\, 1_{Y_0}) = \bar{i}(\sup_{n\in\mathbb{N}} {}^\circ(e_n \wedge g)\, 1_{Y_0}) \\ &= \lim_{n\to\infty} \bar{i}({}^\circ(e_n\wedge g)\, 1_{Y_0}) \leq \sup_{e\in\mathcal{E}_1} \bar{i}({}^\circ e\, 1_{Y_0}).\end{aligned}$$

□

**19. Example.** We show that in Theorem 8 (ii) the assumption $\bar{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e 1_{Y_0}) < \infty$ cannot be dispensed with:

To this aim we construct an internal content $\nu : \mathcal{R} \to {}^*[0,\infty[$ on a ring $\mathcal{R} \subset P(Y)$ and choose $\mathcal{E}$ and $i = i_\nu$ according to Example 5. We furthermore construct $\mathcal{D} \subset \mathcal{R}$ with cardinality of $\mathcal{D}$ smaller than or equal to $\widehat{S}$, $\mathcal{D}\uparrow$ and $Y_0 \subset Y$ such that

$$\bar{\nu}(D \cap Y_0) = 0 \text{ for } D \in \mathcal{D}, \quad \bar{\nu}(\bigcup_{D\in\mathcal{D}} D \cap Y_0) = \infty.$$

Choosing $\mathcal{E}_1 := \{1_D : D \in \mathcal{D}\}$ we obtain

$$\infty = \bar{i}(\sup_{e\in\mathcal{E}_1} {}^\circ e 1_{Y_0}) \neq \sup_{e\in\mathcal{E}_1} \bar{i}({}^\circ e 1_{Y_0}) = 0.$$

Let $Y := {}^*\mathbb{R} \times {}^*\mathbb{R}$ and denote by $\lambda$ the Lebesgue-measure on the Borel-$\sigma$-algebra $\mathbb{B}$ and by $Z$ the counting measure on $\mathbb{B}$. Then

$$\mu(B) := \int \lambda(B_x) Z(dx) \text{ for } B \in \mathbb{B}\times\mathbb{B}$$

defines a measure on $\mathbb{B}\times\mathbb{B}$. Consider

$$\mathcal{R} = {}^*\{\bigcup_{x\in E} \{x\} \times B : E \subset \mathbb{R} \text{ finite and } B \in \mathbb{B} \text{ bounded}\}.$$

Then $\nu = {}^*\mu|\mathcal{R}$ is an internal content on a ring with values in ${}^*[0,\infty[$. Put $\mathcal{D} = \{\bigcup_{x\in E}\{x\} \times {}^*[-n,n] : E \subset \mathbb{R} \text{ finite}, n \in \mathbb{N}\}$. Then $\mathcal{D}\uparrow$ and the cardinality of $\mathcal{D} \subset \mathcal{R}$ is smaller than or equal to $\widehat{S}$. Put

$$Y_0 := \{(x,y) \in \mathbb{R} \times {}^*\mathbb{R} : y \approx x\}.$$

Then $D \cap Y_0 \subset \{(x,y) \in E \times {}^*\mathbb{R} : y \approx x\}$ for some finite $E \subset \mathbb{R}$ and $\bigcup_{D \in \mathcal{D}} D \cap Y_0 = \{(x,y) \in \mathbb{R} \times {}^*\mathbb{R} : y \approx x\} = Y_0$.

Now it suffices to show:

$$(1) \qquad \overline{\nu}(\{(x,y) : x \approx y \in {}^*\mathbb{R}\}) = 0 \text{ for all } x \in \mathbb{R},$$

$$(2) \qquad \overline{\nu}(Y_0) = \infty.$$

Ad (1): We have $\{(x,y) : x \approx y \in {}^*\mathbb{R}\} \subset \{x\} \times {}^*[x - \frac{1}{n}, x + \frac{1}{n}]$ for all $n \in \mathbb{N}$. Then $\overline{\nu}(\{x\} \times {}^*[x - 1/n, x + 1/n]) = st^*\mu(\{x\} \times {}^*[x - \frac{1}{n}, x + \frac{1}{n}]) = \mu(\{x\} \times [x - \frac{1}{n}, x + \frac{1}{n}]) = \frac{2}{n} \to_{n \to \infty} 0$. This proves (1).

Ad(2): Let $B \in \mathcal{R}$ with $Y_0 \subset B$ be given. We have to show

$$(3) \qquad {}^\circ\nu(B) = \infty.$$

For $x \in \mathbb{R}$ we have $m(x) := \{y \in {}^*\mathbb{R} : y \approx x\} \subset B_x$. As $B_x$ is internal, there exists for each $x \in \mathbb{R}$ — according to the permanence principle — $n_x \in \mathbb{N}$ with

$$(4) \qquad {}^*[x - \frac{1}{n_x}, x + \frac{1}{n_x}] \subset B_x.$$

Hence there exists $n_0 \in \mathbb{N}$ such that

$$I_0 = \{x \in \mathbb{R} : n_x = n_0\} \text{ is uncountable.}$$

Therefore according to (4)

$$(5) \qquad \biguplus_{x \in I_0} \{x\} \times {}^*[x - \frac{1}{n_0}, x + \frac{1}{n_0}] \subset B.$$

Thus for each finite $E \subset I_0$ we have

$$ {}^\circ\nu(B) \geq \sum_{x \in E} {}^\circ\nu(\{x\} \times {}^*[x - 1/n_0, x + 1/n_0]) = |E| \, \frac{2}{n_0}.$$

As this holds for each finite $E \subset I_0$, we obtain (3).

The following example shows that the results of Theorem 4 cannot be improved. We consider a set function $\nu$ as in Example 5.

**20. Example.** (i) Let $Y := \{1, 2, 3\}$. Put $\mathcal{R} := \{\emptyset, \{1, 2\}, \{3\}, Y\}$. Let $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ and put

$$\nu(\emptyset) = 0, \ \nu(\{1, 2\}) = \nu(\{3\}) = h, \ \nu(Y) = 2h.$$

Then $\nu : \mathcal{R} \to {}^*[0, \infty[$ is an internal content with

$$\mathcal{M}(\underline{\nu}) = \{\emptyset, \{3\}, \{1, 2\}, Y\} \text{ and } \mathcal{M}(\overline{\nu}) = \mathcal{P}(Y).$$

As $\mathcal{M}_0 = \{\emptyset\}$ we have

$$\underline{\nu}|\mathcal{M}(\underline{\nu}) \text{ is not a saturated measure;}$$

$$\mathcal{M}(\underline{\nu}) \underset{\neq}{\subset} \mathcal{M}(\overline{\nu}).$$

(ii) Let $Y \neq \emptyset$, $\mathcal{R} := \{\emptyset\}$ and $\nu(\emptyset) = 0$. Then $\nu$ is an internal content where $\underline{\nu}(A) = 0$ for each $A \subset Y$ and $\overline{\nu}(A) = \infty$ for each $\emptyset \neq A \subset Y$. Therefore $\mathcal{M}(\underline{\nu}) = \mathcal{M}(\overline{\nu}) = \mathcal{P}(Y)$ and $\mathcal{M}_0 = \{\emptyset\}$. Hence

$$\mathcal{M}_0 \underset{\neq}{\subset} \{A \in \mathcal{M}(\underline{\nu}) : \underline{\nu}(A) \in \mathbb{R}\} (= \mathcal{P}(Y))$$

and therefore

$$\mathcal{L}(i_\nu) \underset{\neq}{\subset} \{f \in \mathbb{R}^Y : f \text{ is } \underline{i_\nu}|\mathcal{M}(\underline{i_\nu})\text{-Lebesgue-integrable}\}.$$

## References

[1] Aldaz, J. M., *A modified functional approach to nonstandard measure theory*, Caribb. J. Math. Comput. Sci. **3** (1993), 17–20. MR96g:28026

[2] Cutland, N., *Nonstandard Analysis and its Applications*, London Mathematical Society Student Texts, 10, Cambridge University Press, Cambridge (1988). MR89m:03060

[3] Floret, K., *Maß- und Integrationstheorie*, B. G. Teubner, Stuttgart, 1981. MR82m:28001

[4] Hurd, A. and Loeb, P., *An introduction to nonstandard real analysis*, Academic Press, Orlando, Tokyo (1985). MR87d:03184

[5] Landers, D. and Rogge, L., *Universal Loeb-measurability of sets and of the standard part map with applications*, Trans. Amer. Math. Soc. **304** (1987), 229–243. MR89d:28015

[6] Landers, D. and Rogge, L., *Nonstandard methods for families of $\tau$-smooth probability measures*, Proc. Amer. Math. Soc. **103** (1988), 1152–1156. MR89j:28007

[7] Landers, D. and Rogge, L., *Nichtstandard Analysis*, Springer-Verlag, Berlin, New York, Tokyo, 1994. MR95i:03140

[8] Loeb, P., *Conversion from nonstandard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122. MR52:10980

[9] Loeb, P., *Weak limits of measures and the standard part map*, Proc. Amer. Math. Soc. **77** (1979), 128–135. MR80i:28020

[10] Loeb, P., *A functional approach to nonstandard measure theory*, Amer. Math. Soc. Contemporary Math. 26 (1984), 251–261. MR86b:28026

[11] Loeb, P., *A nonstandard functional approach to Fubini's theorem*, Proc. Amer. Math. Soc. **93** (1985), 343–346. MR86f:28026

[12] Pfeffer, W., *Integrals and measures*, Marcel Dekker, Inc., New York, Basel, 1977. MR57:573

[13] Sommers, U., *Theorie unendlicher Loeb-Maße*, Diplomarbeit, Duisburg, 1998.

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